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Matrices

①

Properties of determinants:

(i) If two rows or columns of matrix are identical, then the determinant is zero

$$\Delta = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{vmatrix} = 0$$

(ii) If two rows or columns of matrix are interchanged then the sign of determinant changes.

$$\Delta = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} \quad -\Delta = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 1 \end{vmatrix}$$

(iii) If three rows or columns of matrix are interchanged then the sign of determinant is unaltered.

$$\Delta = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

(iv) In the determinant of matrix if any column containing sum or difference of two elements then it can be split into sum or difference of two determinants.

$$\begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Delta = ad - bc$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

TATONORU MAYO

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$$\Delta = a_{11} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Operations for solving matrix are always less than one the order of matrix.

Upper and lower triangular matrix:

If all the elements above principal diagonal are zero then its said to be lower triangular matrix and if all elements below the principal diagonal are zero it is upper triangular matrix.

$$\text{Lower triangular matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix} = 18$$

$$\text{Upper triangular matrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = 24$$

Note:

If is matrix is either lower triangular or upper triangular determinant is product of principal diagonal elements.

Q. Find determinant of matrix

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

(2)

$$R_3 \rightarrow R_3 - R_2 \quad \text{and} \quad R_2 \rightarrow R_2 - R_1$$

$$\begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)$$

(A)

$$|A| = |A^T|$$

Q. Find determinant of

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \quad \text{and} \quad R_3 \rightarrow R_3 - R_1$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 0 \\ 0 & 0 & b \end{vmatrix} = ab$$

Standard results :-

$$(A) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 6 \end{vmatrix} = 20 \quad (B) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix} = abc$$

Q. find the determinant of

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

$$R_1 \rightarrow \frac{R_1}{a}, \quad R_2 \rightarrow \frac{R_2}{b}, \quad R_3 \rightarrow \frac{R_3}{c}$$

$$= abc \begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$R_1 \rightarrow R_1 + (R_2 + R_3)$$

$$= abc \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$C_2 \rightarrow C_2 - C_1, \quad C_3 \rightarrow C_3 - C_1$$

$$= (abc) \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= (abc) \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

standard result:

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd \left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

$$\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = 1 \times 1 \times 1 \times 1 \left(1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1}\right) = 5$$

Q. Find determinant:

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ -3 & 1 & 1 \end{vmatrix}$$

$$\Delta = 0(2-1) - 1(1-9) + 2(1-6) \\ = -2$$

or

$$R_3 \rightarrow R_3 - 3R_2$$

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & -5 & -8 \end{vmatrix}$$

$$\Delta = 0 - 1(-8+10) - \\ = -2$$

or

$$\begin{array}{ccccccc} & & 9 & & 2 & & 4 \\ & 0 & \swarrow & 1 & \searrow & 2 & \swarrow & 0 \\ & & 1 & & 2 & & 1 & & 2 \\ & 12 & \swarrow & 0 & \searrow & 1 & \swarrow & 3 & \searrow & 1 \end{array}$$

$$\Delta = 0 + 9 + 2 - 12 - 0 - 1$$

$\Delta = \sum$ product of 1st diagonal elements

$\sum$ product of 2nd diagonal elements.

$$(1) \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 2 & 2 \end{vmatrix} = 1 + 8 + 8 - 4 - 4 - 4 = 4$$

$$(2) \begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 & 1 & 2 \\ 3 & 1 & 4 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{vmatrix} = 2 + 8 + 15 - 5 - 4 - 12 = 4$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj. } A}{\Delta}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= 1 \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

Inverse matrix exists only for non-singular matrix.

Adjoint of Higher order matrix is the transpose of the co-factor matrix.

Minor of element:

The minor of an element in square matrix is the determinant of square sub matrix in which the row and the column of particular element lies to be deleted.

$$A = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$$

$$\text{Minor of } a_{12} = \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = (1-9) = -8$$

$$\begin{aligned} \text{Co-factor of an element} &= (-1)^{i+j} \text{ Minor of element} \\ &= (-1)^{1+2} \times (-8) \\ &= 8 \end{aligned}$$

[i - no. of row & j - no. of column.]

Shortcut :-

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\Delta = 0 + 9 + 6 - 12 - 0 - 1 = 2$$

0	1	2	0	1	→ Remove
1	2	-7	3	8	1
3	3	1	3	3	3
0	1	2	0	1	
3	2	3	1	2	

Remove

$$A^{-1} = \frac{+1}{2} \begin{bmatrix} -7 & 5 & -1 \\ 8 & -6 & 2 \\ -3 & 3 & -1 \end{bmatrix}$$

1) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$\Delta = 1 + 8 + 8 + -4 -4 -4 = 4$$

1	2	2	1	2
2	1	2	2	1
2	2	1	2	2
1	2	2	1	2
2	1	2	2	1

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$(2) \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\Delta = 3 + 0 - 4 + 0 + 2 + 2 = 9$$

$$\begin{array}{c|cccc} 1 & 2 & -2 & 1 & 2 \\ \hline -1 & 3 & 0 & -1 & 3 \\ 0 & -2 & 1 & 0 & -2 \\ 1 & 2 & -2 & 1 & 2 \\ -1 & 3 & 0 & -1 & 3 \end{array}$$

$$A^{-1} = \frac{-1}{9} \begin{bmatrix} 5 & 2 & +5 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Delta = -1$$

$$\begin{array}{c|cccc} 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{array}$$

$$A^{-1} = -1 \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of matrix :-

If all minors of order $(r+1)$ are zero's, but there is at least one non-zero minor of order r , if exists, is called the Rank of the matrix and is indicated by $\rho(A)=r$. ⑤

Properties of rank :-

(i) If A is null matrix or zero matrix then rank of matrix is zero.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A)=0$$

(ii) If A is a non-zero matrix then rank of A is greater than or equal to one.

$$\rho(A) \geq 1$$

(iii) If I be the unit matrix of order $n \times n$ then rank of I_n is one n .

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \rho(A)=2$$

* (iv) If A be the matrix of order $m \times n$, then rank of matrix A is less than or equal to minimum of m and n .

Q. Find the ranks of following Matrices :

$$(1) \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 5 & 7 \end{bmatrix}_{2 \times 4}$$

$$\rho(A) \leq 2.$$

$$\rho(A)=2$$

(No sub matrix has determinant zero)

$$(2) \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}_{2 \times 3}$$

$$\rho(A) \leq 2.$$

$$\rho(A)=1$$

square sub-matrix are having determinant = 0

$$(1 \times 4 - 2 \times 2) = 0 \quad \& \quad (6 \times 2) - (4 \times 3) = 0$$

$$(3) \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}_{3 \times 3}$$

$$\rho(A) \leq 3$$

$$R_3 \rightarrow R_3 - (R_1 + R_2)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}_{3 \times 3}$$

$$\rho < 3$$

$$\rho(A) = 2$$

(Non-singular square sub-matrix)

$$(4) \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\rho(A) \leq 3$$

(from order of matrix)

$$\rho(A) < 3$$

$$(|A|_{3 \times 3} = 0)$$

$$\rho(A) = 2$$

(non-singular square sub-matrix)

$$(5) \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 8 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}_{3 \times 4}$$

$$R_2 \rightarrow R_2 + R_1, \quad R_3 \rightarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & -1 & 3 & 6 \\ 2 & 7 & 0 & 2 \\ 4 & 4 & 0 & 5 \end{bmatrix}$$

$$\therefore \rho(A) \leq 3$$

$$\rho(A) = 3$$

(non-zero determinant of 3×3 matrix exist)

$$6) \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 3 \end{bmatrix}_{3 \times 4}$$

$$\rho(A) \leq 3$$

$$\rho(A) < 3$$

$$\rho(A) = 2$$

$$7) \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}_{4 \times 4}$$

$$R_4 \rightarrow R_4 - (R_1 + R_2 + R_3)$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) \leq 4$$

$$\rho(A) < 4$$

$$R_2 \rightarrow R_2 - 2R_1$$

(6)

$$\begin{bmatrix} -1 & 2 & 3 & -0 \\ 0 & 0 & -3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\delta(A) = 3$ (non-zero determinant of 3×3 matrix exists)

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}_{3 \times 3}$$

$$\rho(A) \leq 3$$

$$\delta(A) < 3 \quad (3 \times 3 \text{ matrices is zero})$$

$$\delta(A) < 2 \quad (\text{all } 2 \times 2 \text{ matrices are zero})$$

$$\therefore \delta(A) = 1$$

$$\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 5 & 9 & 7 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - (R_1 + R_3)$$

$$R_3 \rightarrow R_3 - (R_1 + R_2)$$

$$\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\delta(A) \leq 4$$

$$\delta(A) < 4 \quad (4 \times 4 \text{-Matrix has 0 deter})$$

$$\rho(A) < 3 \quad (\text{determinant of } 3 \times 3 \text{ is zero})$$

$$\therefore \delta(A) = 2$$

The consistency and inconsistency of system of eqn:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

coefficient matrix, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

constant matrix, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

variable matrix, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Augmented matrix, $(AB) = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$

Any system is said to be consistent system, if it has solution. Inconsistent system has no solution

(i) If $\rho(AB) = \rho(A) = \text{no. of unknowns}$, then the system is said to be consistent and having unique solution.

(ii) If $\rho(AB) = \rho(A) < \text{no. of unknowns}$ then the system is said to be consistent and having more than one solution or infinite no. of solutions.

(iii) If $\rho(AB) \neq \rho(A)$, then system is said to be inconsistent and has no solution.

Q. Find the no. of solutions for system of equations.

$$x + y + z = 3$$

$$x + 2y + 3z = 4$$

$$x + 4y + 9z = 6$$

Note:

(i) If Augmented matrix (AB) is rectangular matrix then last column must be excluded. (Because it will give the rank of (AB) only & we will have to calculate rank of (A) separately.

(ii) If (AB) is square matrix then last column must be included.

(iii) Last column excluded, then represents ranks of both (A) and (AB)

(iv) If last column included gives rank of (AB) only. ⑦

$$(AB) = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{bmatrix}_{3 \times 4}$$

$$R_2 \rightarrow R_2 - R_1, \quad R_3 \rightarrow R_3 - R_1$$

$$= \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{bmatrix}$$

$$\rho(AB) = \rho(A) = 3 \quad (\text{Non-zero determinant of } 3 \times 3 \text{ matrix exists})$$

$\therefore$ The system of equations is consistent and have unique solution.

$$2. \quad x - 2y + 3z = 2$$

$$2x - 3z = 3$$

$$x + y + z = 0$$

$$(AB) = \begin{bmatrix} 1 & -2 & 3 & 2 \\ 1 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{3 \times 4}$$

$$R_1 \rightarrow R_1 + 2R_2$$

$$= \begin{bmatrix} 3 & 0 & 5 & 2 \\ 1 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\rho(AB) = \rho(A) = 3$$

System is consistent and has unique solutions.

$$3x + 3y + 2z = 1$$

$$x + 2y = 4$$

$$10y + 3z = -2$$

$$2x - 3y + z = -5$$

$$AB = \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & 1 & -5 \end{bmatrix}$$

Give preference to the equation with coefficient of x as '1' to write in 1st row.

$$AB = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & -5 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1, \quad R_4 \rightarrow R_4 - 2R_1$$

$$AB = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$

← To find determinant make two zeros in 3x3 matrix.

$$R_2 \rightarrow R_2 + 2R_4, \quad R_3 \rightarrow R_3 + 3R_4$$

$$AB = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -17 & 0 & -17 \\ 0 & -11 & 0 & -11 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2 \quad \& \quad R_2 \leftrightarrow R_4$$

$$AB = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -11 & 0 & -11 \\ 0 & -7 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(-17) \times (-11)$ (common out)

$$\delta(AB) = \delta(A) = 3 = 3$$

$$4x - 2y + 6z = 8$$

$$x + y - 3z = -1$$

$$15x - 3y + 8z = 21$$

$$(AB) = \begin{bmatrix} 1 & 1 & -3 & -1 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 3 & 7 \end{bmatrix}_{3 \times 4}$$

$$R_2 \rightarrow R_2 + R_1 \quad \& \quad R_3 \rightarrow R_3 + R_1$$

$$= \begin{bmatrix} 1 & 1 & -3 & -1 \\ 3 & 0 & 0 & 3 \\ 6 & 0 & 0 & 6 \end{bmatrix}$$

— (i)

$$\rho(AB) = \rho(A) = 2 < 3$$

The system is consistent and have infinite no. of solutions.

(i) $X = A^{-1} \cdot B$

(ii) Crammer's rule

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta}$$

(iii) Gauss-seidal

$$\begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{bmatrix}$$

from — (i)

$$x + y - 3z = -1$$

$$3x = 3$$

$$\therefore x = 1$$

$$6x = 6$$

$$x = 1$$

$$y - 3z = -2$$

if constant is zero $\uparrow$

$$y - 3z = 0$$

$$y = 3z$$

$$\frac{y}{3} = \frac{z}{1}$$

This is called symmetrical method of solving.

∴ We are discussing consistency of real system of equations solutions should be real numbers.

Let $z = k$ where $k \in \text{Real numbers}$.

$$y - 3k = -2$$

$$\therefore y = 3k - 2$$

$$\therefore z = k$$

$$\therefore x = 1$$

For the different values of k , we get different y values
Thus, infinite number of solutions exist.

Q. Check consistency of

$$2x - y + z = 4$$

$$3x - y + z = 6$$

$$4x - y + z = 7$$

$$-x + y + z = 9$$

$$(AB) = \begin{bmatrix} -1 & 1 & -1 & 9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + R_1 \quad R_3 \rightarrow R_3 + R_1 \quad R_4 \rightarrow R_4 + R_1$$

$$= \begin{bmatrix} -1 & 1 & -1 & 9 \\ 1 & 0 & 0 & 13 \\ 2 & 0 & 0 & 15 \\ 3 & 0 & 0 & 16 \end{bmatrix}$$

Always keep zero's either at 1st row or column or last row or column.

Here, don't bring last column inside (it represents B)

$$C_1 \leftrightarrow C_2$$

$$= \left[\begin{array}{c|ccc} 1 & -1 & -1 & 9 \\ 0 & 1 & 0 & 13 \\ 0 & 2 & 0 & 15 \\ 0 & 3 & 0 & 16 \end{array} \right]$$

$$\text{Here } \rho(AB) = 4, \quad \rho(A) = 3$$

The system of equations is inconsistent.

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Q. For what values of λ and μ does the system of equations given below will be having.

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu.$$

(a) No solution

(b) Unique solution

(c) More than one number of solutions.

Case I:

$$(AB) = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix}$$

Case I: If $\lambda - 3 = 0$ & $\mu - 10 \neq 0$

$$\rho(AB) = 3, \quad \rho(A) = 2$$

No solution exists if $\lambda = 3$ & $\mu - 10 \neq 0$

Case II: If $\lambda - 3 = 0$ & $\mu - 10 = 0$

$$\rho(AB) = \rho(A) = 2 < 3 \text{ no. of unknowns}$$

More than one solution exist if $\lambda = 3, \mu = 10$

Case III: If $\lambda - 3 \neq 0$

$$\rho(AB) = \rho(A) = 3 = \text{no. of unknowns.}$$

unique solution exists, if $\lambda \neq 3$ & μ is any real number.

Eigen values and Eigen vectors:

Characteristic equation:

Let 'A' be the square matrix of order 'n x n'. I be the unit matrix of order 'n x n'. then $|A - \lambda I| = 0$, is called the characteristic equation. where λ is a parameter.

The roots of characteristic equation are called the characteristic roots or Latent roots or Eigen values or proper values.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{or} \quad [x_1, x_2, \dots, x_n]^T$$

which satisfies the characteristic equation $[A - \lambda I]X = 0$ is called corresponding Eigen vector of the matrix.

$$[A - \lambda I] \cdot X = 0$$

Note:

(i) The sum of the eigen values of any matrix is equal to sum of the elements of its principal diagonal.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\text{Trace of } A = 1 + 5 + 1$$

$$\text{Tr. (A)} = 7 = (\text{sum of eigen values})$$

(ii) The product of Eigen values of any matrix is equal to its determinant.

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

options: (a) 2.5

(b) 3.4

(c) 6.1

(d) 2.3

characteristic roots of matrix = ?

Sum of diagonals = sum of eigen values

$$= (5 + 2) = 7$$

Product of characteristic roots =

$$\text{determinant} = (10 - 4)$$

$$= 6$$

(iii) The eigen values of symmetric matrix ($A^T = A$) (10)
are purely real.

$$S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$S^T = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

(iv) The eigen values of skew symmetric matrix, i.e.
($A^T = -A$) are either purely imaginary or may be zero

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -A \text{ i.e. skew symmetric}$$

sum of eigen values = 0.

value of determinant = 1

(a) -1, 0

(b) 0, 1

(c) 1, -1

(d) $i, -i$

Ans: (d) $i, -i$ are characteristic roots of matrix

(v) If the matrix is either lower triangular or upper triangular then principal diagonal elements are called the eigen values.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\lambda = 1, 3, 6$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\lambda = 1, -4, 5$$

(vi) After interchanging the rows or columns or after certain operations on matrix, if we obtain lower or upper triangular matrix, we may not necessarily get the values of eigen vector on principal diagonal.

Sunday
22nd September 2013

If λ is eigen value of A then,

$$\begin{array}{ccc} \lambda^2 & \text{is eigen value of} & A^2 \\ \lambda^3 & \text{---} & A^3 \\ \lambda^4 & \text{---} & A^4 \\ \vdots & & \\ \lambda^n & \text{---} & A^n \\ \frac{1}{\lambda} & \text{---} & A^{-1} \end{array}$$

e.g. If $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ with $\lambda = 1, 5$

$$\begin{aligned} S^2 &= \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix} \end{aligned}$$

$$\therefore \lambda = 1^2, 5^2 = 1, 25$$

Q If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ Then eigen values of A^3 are.

$$\begin{array}{l} \lambda \rightarrow A \\ \lambda^3 \rightarrow A^3 \end{array}$$

(a) 9, 1, (b) 5, 4, (c) 16, -1 (d) 27, -1

(d) $27, -1 \Rightarrow (3)^2, (-1)^3$

$$\therefore \lambda^3 = 27, -1$$

Q. Find the eigen values and corresponding eigen vectors of the matrix.

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

(11)

Characteristic equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$$\lambda = 1, 6$$

Eigen vector for $\lambda = 1$.

$$[A - \lambda I] \cdot X = 0$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1 + 4x_2 = 0$$

$$x_1 + x_2 = 0$$

$$x_1 = -x_2$$

$$\frac{x_1}{x_2} = -1$$

$$\frac{x_1}{1} = \frac{x_2}{-1}$$

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & -1 \end{bmatrix}^T$$

Also

$$\frac{x_1}{-1} = \frac{x_2}{1}$$

$$X = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 1 \end{bmatrix}^T \text{ (usually not in options given)}$$

For $\lambda = 6$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-x_1 + 4x_2 = 0$$

$$x_1 - 4x_2 = 0$$

$$x_1 = 4x_2$$

$$\frac{x_1}{4} = \frac{x_2}{1}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ or } [4 \ 1]^T$$

Normalised eigen vector

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1^2 + (-1)^2}} \\ \frac{-1}{\sqrt{1^2 + (-1)^2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{4}{\sqrt{4^2 + 1^2}} \\ \frac{1}{\sqrt{4^2 + 1^2}} \end{bmatrix} = \begin{bmatrix} \frac{4}{\sqrt{17}} \\ \frac{1}{\sqrt{17}} \end{bmatrix}$$

Q. Find the characteristic roots of matrix

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

(a) 3, 7, 8

(b) 2, 2, 14

(c) 0, 3, 15

(d) 1, 4, 9

sum = 18

18

18

14

$$\Delta = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 168 + 48 - 48 - 28 - 128 - 108$$

$$= 0$$

(c) 0, 3, 15

(1) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ Eigen values are

(12)

ca) 1, 2, 3

cb) 0, 2, 4

cc) 1, 1, 4

cd) 2, 2, 2

Sum of eigen values = $1 + 2 + 3 = 6$

Product of eigen values = $\Delta = 6$

Eigen values are ca) 1, 2, 3

Q. The eigen vector pair of matrix $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

cc) $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

cb) $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

cd) $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 9 - 16 = 0$$

$$\lambda^2 = 25$$

$$\lambda = \pm 5$$

For $\lambda = +5$

$$\begin{bmatrix} 3+5 & 4 \\ 4 & -3-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-2x_1 + 4x_2 = 0$$

$$\frac{x_1}{2} = \frac{x_2}{1}$$

$$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda = -5$

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$8x_1 + 4x_2 = 0$$

$$\frac{x_1}{x_2} = \frac{1}{-2}$$

$$x_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Eigen values of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$

$\therefore$ Eigen values = $\lambda = (1, -4, 7)$

For $\lambda = 1$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$2x_2 + 3x_3 = 0^*$$

$$-5x_2 + 2x_3 = 0^*$$

$$6x_3 = 0$$

If in eigen vector equation if two unknowns have two different equations, then those two unknowns have value zero.

If two unknown in the eigen vector equation are zero (for 3×3 matrix) then 3rd unknown is always non zero.

(It may have all values other than zero i.e. infinity)

$\therefore$ Eigen vector for $\lambda = 1$

$$[A - \lambda I] [x] = 0$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad x_4 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda = -4$

$$\begin{bmatrix} 5 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$5x_1 + 2x_2 + 3x_3 = 0$$

$$2x_3 = 0$$

$$x_3 = 0$$

$$5x_1 + 2x_2 = 0$$

$$\frac{x_1}{2} = \frac{x_2}{-5}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}$$

(13)

for $\lambda = 7$

$$\begin{bmatrix} -6 & 2 & 3 \\ 0 & -11 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-6x_1 + 2x_2 + 3x_3 = 0$$

$$-11x_2 + 2x_3 = 0$$

$$-2x_3 = -11x_2$$

$$\therefore \frac{x_2}{-2} = \frac{x_3}{-11}$$

$$\frac{x_2}{2} = \frac{x_3}{11}$$

$$x_3 = 37/6$$

$$X = \begin{bmatrix} 27/6 \\ 2 \\ 11 \end{bmatrix}$$

Q. An eigen vector of matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

(a) $[-1 \ 1 \ 1]^T$ (b) $[1 \ 2 \ 1]^T$ (c) $[1 \ -1 \ 2]^T$ (d) $[2 \ 1 \ -1]^T$

The eigen values are (1, 2, 3) (Upper triangular matrix)

But to find set of eigen vector for particular eigen value, start with least value (i.e. $\lambda = 1$)

$\therefore$ for $\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\therefore x_2 = 0$ not in any option

For $\lambda = 2$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$2x_3 = 0 \quad x_3 = 0$ not in option

For $A=3$

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-2x_1 + x_2 + 0x_3 = 0$$

$$-2x_1 = -x_2$$

$$\frac{x_1}{1} = \frac{x_2}{2}$$

$$\text{Eigen vector} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \text{or} \quad [1 \ 2 \ 0]^T$$

Q. A real $n \times n$ matrix $A = [a_{ij}]$ is given as follow:

$$a_{ij} = i, \quad \text{if } i=j$$

$$= 0 \quad \text{if } i \neq j$$

then sum of all eigen values...

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\text{Sum of Eigen values} = 1 + 2 + 3 + \dots + n$$

$$= \frac{n(n+1)}{2}$$

Cayley-Hamilton theorem:

(14)

Every square matrix satisfies its own characteristic equation.

If $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

characteristic equation

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

According to theorem,

$$A^2 - 7A + 6I = 0$$

I- multiplicative identity.

Proof: —

$$\text{L.H.S.} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - 7 \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 35 & 28 \\ 7 & 14 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= 0$$

Application:

$$6I = 7A - A^2$$

$$6I \cdot A^{-1} = 7A \cdot A^{-1} - A^2 \cdot A^{-1}$$

$$6A^{-1} = 7I - A$$

$$= \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$$

If M be 3×3 matrix. & its characteristic equation $\lambda^3 + 1 = 0$
 then M^{-1} is

(a) M

(b) $-M^2$

(c) M^3

(d) $-M^4$

$$M^3 + I = 0$$

$$M^3 + I = 0$$

$$I = -M^3$$

$$I \cdot M^{-1} = -M^2 \cdot M \cdot M^{-1}$$

$$M^{-1} = -M^2$$

For $P_{3 \times 3}$ if characteristic equation is $\lambda^3 + \lambda^2 + 2\lambda + 1 = 0$

(a) $P^2 + P + I$

(b) $P^2 + P + 2I$

(c) $-(P^2 + P + 2I)$

(d) $-(P^2 + P + I)$

$$P^3 + P^2 + 2P + I = 0$$

$$I \cdot P^{-1} = -(P^3 \cdot P^{-1} + P^2 \cdot P^{-1} + 2P \cdot P^{-1} + I \cdot P^{-1})$$

$$P^{-1} = -(P^2 + P + 2I)$$

If $\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$ then line passes through,

(a) $(0, 0)$

(b) $(3, 4)$

(c) $(4, 3)$

(d) $(4, 4)$

$$\Delta = \begin{vmatrix} x & 2 & 4 & x & 2 \\ y & 8 & 0 & y & 8 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

$$= 8x + 4y - 32 - 2y$$

$$= 8x + 2y - 32$$

(b) $(3, 4)$ satisfies the above equation.

$A_{3 \times 4}$ real matrix and $AX = B$ is an inconsistent system of equations then highest possible rank of A is (15)

- (a) 1 (b) 2 (c) 3 (d) 4

∴ We add column to A when we add B in matrix A

∴ If A is 3×4 $AB = 3 \times 5$

∴ Max. possible rank $\rho(AB) = 3$

But given that, system is inconsistent

$$\rho(AB) \neq \rho(A)$$

In no case $\rho(A) > \rho(AB)$

$$\rho(A) = 2$$

The value of 'a' for which the following set of equations have a non-trivial solution is

$$y + 2z = 0$$

$$2x + y + z = 0$$

$$ax + 2y + 0z = 0$$

- (a) 2 (b) 4 (c) 8 (d) None

For the system of equations $AX = 0$, $X = 0$ is always a solution and that solution is called a trivial solution.

For a non-trivial solution, determinant of coefficient matrix is always zero.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 1 \\ a & 2 & 0 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 0 & 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 2 & 1 \\ a & 2 & 0 & a & 2 \end{vmatrix}$$

$$= a + 8 - 2a$$

$$0 = 8 - a$$

$$a = 8$$

Note:

- ci) $\Delta \neq 0, X = 0 \Rightarrow$ trivial solution
cii) $\Delta = 0, X \neq 0 \Rightarrow$ non-trivial solution.

Given an orthogonal matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\therefore [A \cdot A^T]^{-1} = ?$$

A matrix A is said to be an orthogonal matrix if

$$A \cdot A^T = I$$

$$\therefore [A \cdot A^T]^{-1} = I^{-1} \\ = I$$

The inverse of the matrix

$$A = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

We know that,

Inverse of diagonal matrix = inverse of each element in diagonal.

$$A^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Rank of diagonal matrix is equal to number of non-zero elements in the diagonal.

Q. The rank and nullity of matrix is

$$\begin{bmatrix} 3 & 0 & 0 & 2 \\ 4 & 7 & 5 & 3 \\ 1 & 7 & 2 & 1 \end{bmatrix}$$

$$\begin{aligned} \text{Nullity of matrix} &= \text{No. of columns} - \text{Rank} \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{If } |A| &= 2 & \therefore |A^2| &= |A|^2 \\ & & |A^3| &= |A|^3 \\ & & & \vdots \end{aligned}$$

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

$$\begin{aligned} |A^{-1}| &= \frac{|\text{adj } A|}{|A|} \\ &= \frac{|\text{adj } A|}{|A|^n} \end{aligned}$$

$$\begin{aligned} |\text{adj } A| &= |A^{-1}| \cdot |A|^n \\ &= |A|^{n-1} \end{aligned}$$

$$\begin{aligned} |\text{adj}(\text{adj } A)| &= |\text{adj } A|^{n-1} \\ &= (|A|^{n-1})^{n-1} \\ &= |A|^{(n-1)^2} \end{aligned}$$

Q. If $A_{3 \times 3}$ is real matrix and B is adjoint of A & $|B| = 64$
then $|A| = ?$

$$|B| = |\text{adj } A| = |A|^{n-1}$$

$$64 = |A|^{3-1}$$

$$64 = |A|^2$$

$$|A| = \pm 8$$

If $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -2 & 0 \\ 4 & 2 & -1 \end{bmatrix}$ & $B = 2A^2$ then $|B| = ?$

(a) 16

(b) 32

(c) 64

(d) 128

$$B = 2A^2$$

$$|B| = |2A^2|$$

$$= 2^n |A^2|$$

$$= 2^3 (16)$$

$$= 128$$

This is not modulus
but determinant
n-order of A

If $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix}$ then $a+b = ?$

$$A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6} \begin{bmatrix} 3 & 0.1 \\ 0 & 2 \end{bmatrix}$$

$$a = \frac{0.1}{6} \quad \text{and} \quad b = \frac{2}{6}$$

$$a+b = \frac{2.1}{6}$$

$$= \frac{7}{20}$$

Let A, B, C, D be nxn matrices each with non-zero determinant & if $A \cdot B \cdot C \cdot D = I$ then $B^{-1} = ?$

(a) $D^{-1}C^{-1}A^{-1}$

(b) ADC

(c) CDA

(d) doesn't exist.

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$CABCD)^{-1} = I^{-1}$$

$$D^{-1} \cdot C^{-1} \cdot B^{-1} \cdot A^{-1} = I$$

$$D \cdot D^{-1} \cdot C^{-1} \cdot B^{-1} \cdot A^{-1} = D \cdot I$$

$$C \cdot C^{-1} \cdot B^{-1} \cdot A^{-1} = C \cdot D$$

$$B^{-1} \cdot A \cdot A^{-1} = C \cdot D \cdot A$$

In modern algebra
(matrices)

$$AB \neq BA$$

For what value of a , if any, will the following system of equations i.e.

$$2x + 3y = 4$$

$$x + y + z = 4$$

$$x + 2y - z = a$$

has a solution.

∴ Given that,

$$\rho(CAB) = \rho(CA)$$

$$AB = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 3 & 0 & 4 \\ 1 & 2 & -1 & a \end{bmatrix}_{3 \times 4}$$

$$R_3 \rightarrow R_3 + R_1$$

$$AB = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 4 \\ 2 & 3 & 0 & a+4 \end{bmatrix}$$

Case I: If $(a+4) \neq -4$

$$\rho(CAB) = 3 \quad \& \quad \rho(CA) = 2 \quad (\text{not true here})$$

Case II: If $(a+4) = -4$

$$a = -8$$

For the matrix $M = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$, $M^T = M^{-1}$ then $x = ?$

(a) $-\frac{4}{5}$

(b) $-\frac{3}{5}$

(c) $\frac{3}{5}$

(d) $\frac{4}{5}$

$$M^T = \begin{bmatrix} \frac{3}{5} & x \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

— (i)

$$M^{-1} = \frac{1}{\frac{9}{25} - \frac{4x}{5}} \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$$

— (ii)

For $M^T = M^{-1}$

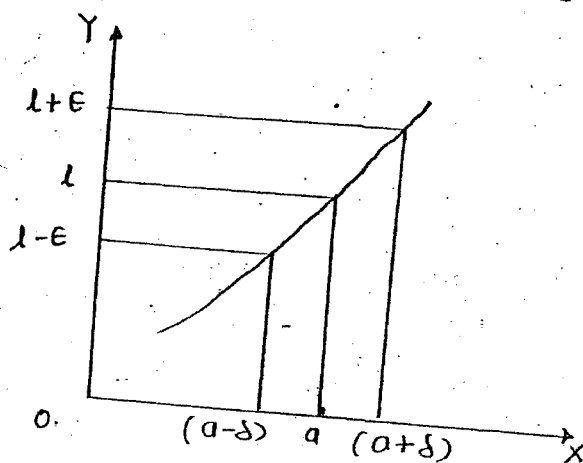
$$x = -\frac{4}{5}$$

"Differential Calculus"

A number L is said to be the limit of function $f(x)$ as $x \rightarrow a$, if for all $\epsilon > 0$ (however small) there exist a $\delta > 0$ such that,

$$|f(x) - L| < \epsilon \quad \forall \quad |x - a| < \delta$$

$$\therefore L - \epsilon < f(x) < L + \epsilon \quad \forall \quad a - \delta < x < a + \delta$$



$$\therefore \lim_{x \rightarrow a} f(x) = L$$

• Left limit:

$$\lim_{x \rightarrow a^-} f(x) = \lim_{h \rightarrow 0} f(a-h), \quad (a-\delta) < x < a$$

• Right limit:

$$\lim_{x \rightarrow a^+} f(x) = \lim_{h \rightarrow 0} f(a+h), \quad a < x < a+\delta$$

Existence of limit

The limit of a function exists if both the left and right limits are existed and are equal

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$$

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$$

$$\lim_{x \rightarrow a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} \cdot a^{m-n}$$

$$\star \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \quad \text{0/0}$$

$$\lim_{x \rightarrow 0} \frac{e^{mx} - 1}{x} = m \quad \checkmark$$

$$\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log a$$

$$\lim_{x \rightarrow 0} \frac{a^x - b^x}{x} = \log\left(\frac{a}{b}\right)$$

$$\lim_{x \rightarrow 0} \frac{(1+x)^{1/x} - e}{x} = 0$$

$$\lim_{x \rightarrow 0} \frac{(1+ax)^{1/x} - e^a}{x} = 0$$

$$\lim_{x \rightarrow 0} \frac{(1+ax)^{b/x} - e^{ab}}{x} = 0$$

$$\lim_{x \rightarrow 0} \frac{(1+ax)^{1/bx} - e^{a/b}}{x} = 0$$

$$\lim_{x \rightarrow 0} \frac{(1+\frac{1}{x})^x - e}{x} = 0$$

→ apply L hospital

$$\frac{0}{0}, \frac{\infty}{\infty}$$

$$2 \sin A \cos B \\ = \sin(A+B) + \sin(A-B)$$

(18)

$$\checkmark \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\checkmark \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos mx}{1 - \cos nx} = \frac{m^2}{n^2}$$

$$\star \star \star \lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{x^2} = \frac{b^2 - a^2}{2}$$

 $\left[\frac{0}{0}, \frac{\infty}{0}, \frac{\infty}{\infty}, \frac{0}{\infty} \right]$ etc are the indeterminate forms
 If $\frac{0}{0}, \frac{\infty}{\infty}$ then apply the L-Hospitality rule.

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

(both $f(x)$ & $g(x)$ are algebraic functions)

$$= \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)}$$

$$\text{Special} \lim_{x \rightarrow a} [f(x)]^{g(x)}$$

$$= \lim_{x \rightarrow a} \frac{g(x) [f(x)-1]}{f(x)}$$

if $f(x) = \text{finite}$

$g(x) = \text{infinite}$

as $x \rightarrow a$

$$\cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos(x+y) \\ \sin(x+y)$$

$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$, if both $f(x)$ and $g(x)$ are algebraic function

Case I:

If the degree of $f(x) > g(x)$ degree, then result is ∞ .

$$\lim_{x \rightarrow \infty} \frac{2x^3 + 3x + 5}{3x^2 + 6x + 7} = \infty$$

Case II:

If the degree of $f(x) < g(x)$, then result is zero.

$$\lim_{x \rightarrow \infty} \frac{2x^2 + 3x + 4}{x^3 + 5} = 0$$

Case III:

If degrees of $f(x)$ and $g(x)$ are equal then.

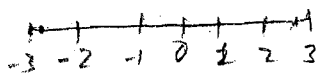
$$\lim_{x \rightarrow \infty} \frac{2x^2 + 3x + 4}{5x^2 + 6x + 7} = \frac{2}{5} = \frac{\text{coeff. of numerator}}{\text{coeff. of denominator}}$$

$$\begin{cases} |x| = x & \text{if } x > 0 \\ |x| = -x & \text{if } x < 0 \end{cases}$$

$[x]$ - step x - The greatest integer n of greater than x

$$[2.99] = 2$$

$$[-2.99] = -3$$



Q. 1. $\lim_{x \rightarrow a} \frac{1}{x-a} =$

(a) 0

(b) ∞

(c) $-\infty$

(d) does not exist

$$\checkmark \lim_{x \rightarrow a^-} f(x) = \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} \frac{1}{a-h-a} = \frac{-1}{h} = \frac{-1}{0} = -\infty$$

$$\checkmark \lim_{x \rightarrow a^+} f(x) = \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} \frac{1}{a+h-a} = \frac{1}{h} = \infty$$

lim does not exist

$$\lim_{x \rightarrow a} f(x) = \lim_{h \rightarrow 0} f(a-h) = \lim_{h \rightarrow 0} \frac{1}{a-h-a} = \lim_{h \rightarrow 0} \frac{-1}{h} = -\infty$$

$LHS \neq RHS$ does not exist

$$2. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x \cdot \sin x}$$

$$L = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x \cdot \sin x}$$

$$= \frac{1 - 0}{0} = \frac{0}{0}$$

Use L-hospitality rule.

$$d(\cos x) = -\sin x$$

$$d(\sin x) = \cos x$$

$$L = \lim_{x \rightarrow 0} \frac{0 + \sin x}{x \cdot \cos x + \sin x (1)}$$

$$= \lim_{x \rightarrow 0} \frac{\cos x}{x(-\sin x) + \cos x (1) + \cos x}$$

$$= \frac{1}{0 + 1 + 1}$$

$$= \frac{1}{2}$$

$$\sin x = 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2}$$

$$1 + \cos x = 2 \cos^2 \frac{x}{2}$$

$$1 - \cos x = 2 \sin^2 \frac{x}{2}$$

Method 2

$$L = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x \cdot \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x \cdot 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2}}$$

$$= \lim_{x \rightarrow 0} \frac{\tan \frac{x}{2}}{\left(\frac{x}{2}\right) \times 2}$$

$$= \frac{1}{2}$$

Method 3

$$L = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x \cdot \sin x} \cdot \frac{(1 + \cos x)}{(1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x \cdot \sin x (1 + \cos x)}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x \cdot \sin x (1 + \cos x)} \\
&= \lim_{x \rightarrow 0} \frac{\sin x}{x (1 + \cos x)} \\
&= \lim_{x \rightarrow 0} \frac{(\sin x / x)}{(1 + \cos x)} \\
&= \frac{1}{1 + 1} \\
&= \frac{1}{2}
\end{aligned}$$

$$(2) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$$

$$= \frac{1 - 1}{0} = \frac{0}{0}$$

$$L = \lim_{x \rightarrow 0} \frac{0 + \sin x}{2x}$$

(L-hospitality)

$$= \frac{1}{2}$$

$$(3) \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin^3 x}$$

$$L = \frac{\tan x - \sin x}{\sin^3 x}$$

If both numerator & denominator are trigonometric function, don't use L-hospitality rule

$$L = \lim_{x \rightarrow 0} \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\cos x \cdot \sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{(1 - \cos x)}{(1 - \cos x)(1 + \cos x) \cdot \cos x}$$

$$= \frac{1}{1 + 1}$$

$$= \frac{1}{2}$$

$$(4) \lim_{x \rightarrow 1} \frac{\cos\left(\frac{\pi x}{2}\right)}{1 - \sqrt{x}}$$

(20)

$$d(\cos ax) = -a \sin ax$$

$$d(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$L = \lim_{x \rightarrow 1} \frac{-\frac{\pi}{2} \sin \frac{\pi x}{2}}{1 - \frac{1}{2\sqrt{x}}}$$

$$= \frac{\pi(1)}{\frac{1}{\sqrt{1}}}$$

$$= \pi$$

$$(5) \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 0$$

$$L = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x} \cdot 1}{1}$$

$$d(\log x) = \frac{1}{x}$$

$$= \frac{1}{1}$$

$$= 1$$

$$(6) \lim_{x \rightarrow 0} \frac{\sqrt{1-x^2} - \sqrt{1+x^2}}{x^2} = \frac{0}{0}$$

$$L = \lim_{x \rightarrow 0} \frac{\frac{1}{2\sqrt{1-x^2}}(-2x) - \frac{1}{2\sqrt{1+x^2}}(2x)}{2x}$$

$$= \lim_{x \rightarrow 0} \frac{-\frac{1}{2\sqrt{1-0}} - \frac{1}{2\sqrt{1+0}}}{1}$$

$$= -\frac{1}{2} - \frac{1}{2}$$

$$= -1$$

$$(7) \lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x}$$

$$\begin{aligned} L &= \lim_{x \rightarrow 0} \frac{(1+x)^{1/3} - (1-x)^{1/3}}{x} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{3} (1+x)^{-2/3} - \frac{1}{3} (1-x)^{-2/3} (-1)}{1} \\ &= \frac{1}{3} \left[(1+0)^{-2/3} + (1)^{-2/3} \right] \\ &= \frac{2}{3} \end{aligned}$$

$$(8) \lim_{x \rightarrow 8} \frac{x^{1/3} - 2}{x - 8}$$

$$\begin{aligned} L &= \lim_{x \rightarrow 8} \frac{x^{1/3} - 8^{1/3}}{(x-8)} = \frac{0}{0} \text{ form} \\ &= \lim_{x \rightarrow 8} \frac{\frac{1}{3} x^{-2/3} - 0}{1} \\ &= \frac{1}{3} (8)^{-2/3} \\ &= \frac{1}{3} (2^3)^{-2/3} \\ &= \frac{1}{3} \times \frac{1}{4} \\ &= \frac{1}{12} \end{aligned}$$

$$(9) \lim_{x \rightarrow 0} (1+2x)^{1/3x}$$

$$\begin{aligned} &= \lim_{x \rightarrow 0} (1+2x)^{1/3x} \\ &= e^{2/3} \end{aligned}$$

$$(10) \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^{2n}$$

$$L = \lim_{n \rightarrow \infty} \left[\left(1 - \frac{1}{n}\right)^n \right]^2$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e^1$$

(2)

$$\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^x = e^{-1}$$

$$\lim_{x \rightarrow \infty} \left(1 - \frac{2}{x}\right)^x = e^{-2}$$

$$L = (e^{-1})^2$$

$$= e^{-2}$$

$$(11) \lim_{x \rightarrow \infty} \left(\frac{x-1}{x-2}\right)^x$$

$$L = \lim_{x \rightarrow \infty} \left(\frac{x-1}{x-2}\right)^x = \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{x \rightarrow \infty} \left[\frac{x \left(1 - \frac{1}{x}\right)}{x \left(1 - \frac{2}{x}\right)} \right]^x$$

$$= \lim_{x \rightarrow \infty} \frac{\left(1 - \frac{1}{x}\right)^x}{\left(1 - \frac{2}{x}\right)^x}$$

$$= \frac{e^{-1}}{e^{-2}}$$

$$= e$$

$$(1) \lim_{n \rightarrow \infty} \frac{1+2+3+\dots+n}{n^2}$$

$$L = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2n^2}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)}{2n}$$

$$= \frac{1}{2}$$

degree of numerator and denominator = 1 (same)

coeff. of num.

coeff. of denominator.

$$(13) \lim_{n \rightarrow \infty} \frac{n(1^3 + 2^3 + \dots + n^3)^2}{(1^2 + 2^2 + \dots + n^2)^3}$$

$$1^2 + 2^2 + 3^2 + \dots = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$L = \lim_{n \rightarrow \infty} \frac{n \left[\frac{n^2(n+1)^2}{4} \right]^2}{\left[\frac{n(n+1)(2n+1)}{6} \right]^3}$$

$$= \lim_{n \rightarrow \infty} \frac{n^5(n+1)^4}{16} \times \frac{216}{n^3(n+1)^3(2n+1)^3}$$

$$= \frac{1 \times 216}{16 \times 1 \times 1 \times 8}$$

$$= \frac{27}{16}$$

$$(14) \lim_{n \rightarrow \infty} \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} \right]$$

$$L = \lim_{n \rightarrow \infty} \left[\sum \frac{1}{n(n+1)} \right]$$

If no standard formula is available & sum is given use

$$S_n = \sum T_n$$

$$L = \lim_{n \rightarrow \infty} \sum \left(\frac{1}{n} - \frac{1}{n+1} \right) \quad \text{— shortcut for partial fraction, here}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{n} + \frac{1}{n+1} \right)$$

$$= 1 - \frac{1}{n+1}$$

$$= 1 - \frac{1}{\infty} = 1 - 0 = 1$$

15. $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{1 - \cos 3x}$

(22)

$$\begin{aligned} L &= \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{1 - \cos 3x} \\ &= \frac{2^2}{3^2} \\ &= \frac{4}{9} \end{aligned}$$

The continuity of a function:

A function $f(x)$ is said to be continuous at $x = a$ if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

otherwise it is said to be a discontinuous function.

Types of discontinuous functions:

(i) Discontinuity of I-type (Jump discontinuity)

A function is said to have discontinuity of I-type if

$$\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x) \quad (\text{limit doesn't exist})$$

(ii) Discontinuity of II-type

A function is said to have discontinuity of II-type if either left limit or right limit or both does not exist.

(iii) Discontinuity of III-type (Removable discontinuity)

A function is said to have discontinuity of III-type if

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) \neq f(a)$$

Note-

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a) \rightarrow \text{cont}$$

Standard continuous functions:

- (i) x^n is continuous function $\forall x$ when $n > 0$ i.e. $(x, x^2, x^3, \dots)$
- (ii) x^n is continuous function $\forall x$ except $x=0$, when $n < 0$
- (iii) $|x|$ is continuous function $\forall x$
- (iv) $\log x$ is continuous function $\forall x$ if $x > 0$
- (v) Every exponential function is continuous (e^x or a^x)
- (vi) $\sin x$ and $\cos x$ are continuous functions for all values of x .
- (vii) $\tan x$ and $\sec x$ are continuous functions $\forall x$, except $\cos x = 0$,
i.e. except, $x = (2n-1) \cdot \frac{\pi}{2}$
- (viii) $\cot x$ and $\operatorname{cosec} x$ are continuous functions $\forall x$ except for
 $\sin x = 0$, i.e. $x = n\pi$
- (ix) Every algebraic function is continuous ($a_0 x^n + a_1 x^{n-1} + \dots + a_n$)
- (x) Every rational function $\left(\frac{f(x)}{g(x)}\right)$ is continuous except $g(x) = 0$

Q. Check the continuity of the function.

$$f(x) = \frac{1}{1 + 2^{1/x}} \quad \text{at } x = 0$$

Left limit

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0-h)$$

$$= \lim_{h \rightarrow 0} \frac{1}{1 + 2^{1/0-h}}$$

$$2^{-1/h} = 2^{-1/0} = 2^{-\infty} = 0$$

$$2^{1/h} = 2^{1/0} = 2^{\infty} = \infty$$

$$L = \lim_{h \rightarrow 0} \frac{1}{1 + 2^{-1/h}}$$

$$= \frac{1}{1 + 0}$$

$$= 1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0+h)$$

$$= \lim_{h \rightarrow 0} \frac{1}{1+2^{1/h}}$$

$$= \lim_{h \rightarrow 0} \frac{1}{1+\infty}$$

$$= \frac{1}{\infty}$$

$$= 0$$

As $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$ - function is discontinuous (1-ly)

(2) $f(x) = \frac{x(e^{1/x}-1)}{(e^{1/x}+1)}$ at $x=0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} \frac{(0-h)(e^{-1/h}-1)}{(e^{-1/h}+1)}$$

$$0 \left(\frac{0}{0} \right) = 0 \quad 0 \left(\frac{\infty}{\infty} \right) \neq 0$$

$$e^{-1/h} = e^{-1/0} = e^{-\infty} = 0$$

$$e^{1/h} = e^{1/0} = e^{\infty} = \infty$$

$$L = \frac{-h(0-1)}{(0+1)} = h$$

$$= 0$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} \frac{(0+h)(e^{1/h}-1)}{(e^{1/h}+1)}$$

$$= \lim_{h \rightarrow 0} \frac{h(e^{1/h}-1)}{(e^{1/h}+1)}$$

$$= 0 \left(\frac{\infty}{\infty} \right)$$

$$= \lim_{h \rightarrow 0} \frac{h \cdot e^{1/h} \left(1 - \frac{1}{e^{1/h}} \right)}{e^{1/h} \left(1 + \frac{1}{e^{1/h}} \right)}$$

$$\begin{aligned}
 &= \frac{h \left(1 - \frac{1}{\infty}\right)}{\left(1 + \frac{1}{\infty}\right)} \\
 &= \frac{h (1 - 0)}{(1 + 0)} \\
 &= 0
 \end{aligned}$$

Function is continuous.

Q. A function $f(x)$ is defined by

$$\begin{aligned}
 f(x) &= 0 & x &\leq 0 \\
 &= \frac{1}{2} - x & 0 < x < \frac{1}{2} \\
 &= \frac{1}{2} & x &= \frac{1}{2} \\
 &= \frac{3}{2} - x & \frac{1}{2} < x < 1 \\
 &= 1 & x &\geq 1
 \end{aligned}$$

which of following is true

- (a) $f(x)$ is continuous at $x=0$
- ✓ (b) $f(x)$ is discontinuous at $x=\frac{1}{2}$
- (c) $f(x)$ is continuous at $x=1$
- (d) All are true.

(a) For continuity at $x=0$

$$\begin{aligned}
 \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^+} f(x) = f(0) \\
 x < 0 & \quad x > 0 & x = 0 \\
 0 \text{ (given)} & \quad \frac{1}{2} - x \text{ (given)} \\
 &= \frac{1}{2} - 0 = \frac{1}{2}
 \end{aligned}$$

Thus not true.

(b) At $x = \frac{1}{2}$

$$\begin{aligned}
 x < \frac{1}{2} & \quad x > \frac{1}{2} & x = \frac{1}{2} \\
 \frac{1}{2} - \frac{1}{2} = 0 & \quad \frac{3}{2} - x \\
 &= \frac{3}{2} - \frac{1}{2} \\
 &= 1
 \end{aligned}$$

∴ Discontinuous at $x = \frac{1}{2}$

$$\begin{aligned}
 f(x) &= -x^2 & x &\leq 0 \\
 &= 5x - 4 & 0 < x &\leq 1 \\
 &= 4x^2 - 3x & 1 < x &\leq 2 \\
 &= 3x + 4 & x &> 2
 \end{aligned}$$

(2)

At $x=0$

$$\begin{aligned}
 x &< 0 \\
 &= -x^2 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 x &> 0 \\
 &= 5(0) - 4 \\
 &= -4
 \end{aligned}$$

— discontinuous

At $x=1$

$$\begin{aligned}
 x &< 1 \\
 &= 5(1) - 4 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 x &> 1 \\
 &= 4(1)^2 - 3(1) \\
 &= 1
 \end{aligned}$$

$$x = 1$$

$$5(1) - 4$$

$$= 1 \text{ — continuous}$$

At $x=2$

$$\begin{aligned}
 x &< 2 \\
 &= 4(2)^2 - 3(2) \\
 &= 8 - 6 \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 x &> 2 \\
 &= 3(2) + 4 \\
 &= 10
 \end{aligned}$$

— discontinuous

Differentiability :→

Monday
23rd September 2013

A function $f(x)$ is said to be differentiable at $x=a$ if

$$\text{Left hand derivative} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{-h}$$

$$\text{Right hand derivative} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

both LHD & RHD exist and are equal

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{-1}{2x\sqrt{x}}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-1}{x^3}$$

$$\frac{d}{dx}(e^{ax}) = a \cdot e^{ax}$$

$$\frac{d}{dx}(a^x) = a^x \cdot \log a$$

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\frac{d}{dx}(x^x) = x^x(1 + \log x)$$

$$\frac{d}{dx}(x^{1/x}) = x^{1/x} \cdot \frac{1}{x^2} (1 - \log x)$$

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

$$\text{If } y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

then

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\text{If } y = x^{x^2}$$

$$\therefore y = x^y$$

$$\log y = y \cdot \log x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\left(\frac{1}{y} - \log x\right) \frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y^2}{x(1 - y \log x)}$$

$$\text{If } x^y = e^{x-y}$$

$$\log x^y = \log e^{x-y}$$

TAT BOSTON MICRO

ASTRONOMY

$$y \log x = (x-y) \cdot \log e$$

$$y + y \log x = x$$

$$y(1 + \log x) = x$$

$$y = \frac{x}{1 + \log x}$$

$$\frac{dy}{dx} = \frac{(1 + \log x) - x \left(0 + \frac{1}{x}\right)}{(1 + \log x)^2}$$

$$= \frac{\log x}{(1 + \log x)^2}$$

Find derivatives of

$$x^m \cdot y^n = a^{m+n}$$

$$x^m \cdot y^n = a^{m+n}$$

$$\log(x^m \cdot y^n) = \log a^{m+n}$$

$$\log x^m + \log y^n = \log a^{m+n}$$

$$m \log x + n \log y = \log a^{m+n}$$

$$m \cdot \frac{1}{x} + n \cdot \frac{1}{y} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-my}{nx}$$

$$x^a \cdot y^b = (x+y)^{a+b}$$

$$\log(x^a \cdot y^b) = \log(x+y)^{a+b}$$

$$a \cdot \log x + b \cdot \log y = (a+b) \cdot \log(x+y)$$

$$a \cdot \frac{1}{x} + b \cdot \frac{1}{y} \frac{dy}{dx} = (a+b) \cdot \frac{1}{(x+y)} \left(1 + \frac{dy}{dx}\right)$$

$$\frac{a}{x} + \frac{b}{y} \cdot \frac{dy}{dx} = \left(\frac{a+b}{x+y}\right) + \left(\frac{a+b}{x+y}\right) \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} \left(\frac{b}{y} - \frac{a+b}{x+y}\right) = \left(\frac{a+b}{x+y}\right) - \frac{a}{x}$$

$$\frac{dy}{dx} \left(\frac{bx - by - ay - by}{y(x+y)}\right) = \frac{ax + bx - ax - ay}{x(x+y)}$$

$$\frac{dy}{dx} = \frac{bx - ay}{x(x+y)} \times \frac{y(x+y)}{bx - ay}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ then $(1-x^2)y_1 - xy =$

(a) 0

(b) 1

(c) y

(d) -y

$$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

$$y^2 (1-x^2) = (\sin^{-1} x)^2$$

$$y^2 (-2x) + (1-x^2) \cdot 2y \cdot y_1 = \frac{2 \sin^{-1} x}{\sqrt{1-x^2}}$$

$$2y (-xy + (1-x^2) \cdot y_1) = \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} = 2y$$

$$(1-x^2) \cdot y_1 - xy = 1$$

If $y = a \cos(\log x) + b \sin(\log x)$ then $x^2 y_2 + x y_1 =$

(a) 0

(b) 1

(c) y

(d) $-y$

(26)

$$y_1 = -a \sin(\log x) \cdot \frac{1}{x} + b \cos(\log x) \cdot \frac{1}{x}$$

$$x y_1 = b \cos(\log x) - a \sin(\log x)$$

$$x^2 y_2 + x y_1 = \frac{-b \sin(\log x) - a \cos(\log x)}{x}$$

$$\begin{aligned} x^2 y_2 + x y_1 &= -[a \cos(\log x) + b \sin(\log x)] \\ &= -y \end{aligned}$$

~~★~~ If $y = \sqrt{\sin x} \sqrt{\sin x} \sqrt{\sin x} \sqrt{\dots \infty}$

$$\frac{dy}{dx} = \frac{f'(x)}{2y-1}$$

$$\frac{dy}{dx} = \frac{\cos x}{2y-1}$$

$$y = \sqrt{\sin x} \Rightarrow y^2 = \sin x$$

$$2y \frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$(2y-1) \frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{(2y-1)}$$

If $y = \sqrt{\log x} \sqrt{\log x} \sqrt{\log x} \dots \infty$

$$\frac{dy}{dx} = \frac{1}{x(2y-1)}$$

If $y = \sqrt{\tan x} \sqrt{\tan x} \sqrt{\tan x} \dots \infty$

$$\frac{dy}{dx} = \frac{\sec^2 x}{2y-1}$$

If $y = \sqrt{x} \sqrt{x} \sqrt{x} \dots \infty$

$$\frac{dy}{dx} = \frac{1}{2y-1}$$

If $\tan^{-1} \left[\frac{(3-x)\sqrt{x}}{1-3x} \right]$ then $\frac{dy}{dx} = ?$ at $x=1$

Whenever the inside function is algebraic, trigonometric substitutions are required.

put $x = \tan^2 \theta \Rightarrow$

$$\sqrt{x} = \sqrt{\tan^2 \theta}$$

$$\theta = \tan^{-1}(\sqrt{x})$$

$$y = \tan^{-1} \left[\frac{(3 - \tan^2 \theta) \cdot \tan \theta}{1 - 3 \tan^2 \theta} \right]$$

$$= \tan^{-1} \left[\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right]$$

$$= \tan^{-1}(\tan 3\theta)$$

$$y = 3\theta$$

$$= 3 \tan^{-1}(\sqrt{x})$$

$$\frac{dy}{dx} = 3 \cdot \frac{1}{(1+x)} \cdot \frac{1}{2\sqrt{x}}$$

At $x=1$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3}{(1+1)} \cdot \frac{1}{2(1)} \\ &= \frac{3}{4} \end{aligned}$$

If $y = \tan^{-1} \left[\frac{\cos x}{1 + \sin x} \right]$ then $\frac{dy}{dx} = ?$

$$y = \tan^{-1} \left[\frac{\sin(\frac{\pi}{2} - x)}{1 + \cos(\frac{\pi}{2} - x)} \right]$$

$$= \tan^{-1} \left[\frac{2 \sin(\frac{\pi}{4} - \frac{x}{2}) \cdot \cos(\frac{\pi}{4} - \frac{x}{2})}{2 \cos^2(\frac{\pi}{4} - \frac{x}{2})} \right]$$

$$= \tan^{-1} \left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \right]$$

$$y = \frac{\pi}{4} - \frac{x}{2}$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

If $x = a \cos^3 \theta$, $y = a \sin^3 \theta$. find $\frac{dy}{dx} =$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{d}{d\theta} (a \sin^3 \theta)}{\frac{d}{d\theta} (a \cos^3 \theta)} \\ &= \frac{3a \cdot \sin^2 \theta \cdot (\cos \theta)}{-3a \cos^2 \theta \cdot (\sin \theta)} \\ &= -\tan \theta \end{aligned}$$

$x = 3 \cos \theta - \cos^3 \theta$, $y = 3 \sin \theta - \sin^3 \theta$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{d}{d\theta} (3 \sin \theta - \sin^3 \theta)}{\frac{d}{d\theta} (3 \cos \theta - \cos^3 \theta)} \\ &= \frac{3 \cos \theta - 3 \sin^2 \theta \cdot \cos \theta}{-3 \sin \theta + 3 \cos^2 \theta \cdot \sin \theta} \\ &= \frac{(1 - \sin^2 \theta) \cdot 3 \cos \theta}{-(1 - \cos^2 \theta) \cdot 3 \sin \theta} \\ &= \frac{\cos^2 \theta \cdot 3 \cos \theta}{-\sin^2 \theta \cdot 3 \sin \theta} \\ &= -\cot^3 \theta \end{aligned}$$

$$x = a(\theta + \sin \theta)$$

$$y = a(1 - \cos \theta)$$

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta} (a(1 - \cos \theta))}{\frac{d}{d\theta} (a(\theta + \sin \theta))}$$

$$= \frac{a [0 + \sin^2 \theta]}{a [1 + \cos \theta]}$$

$$= \frac{2 \cdot 2 \cdot \frac{1}{2} \sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}$$

$$= \frac{1}{2} \tan \frac{\theta}{2}$$

Mean value theorem:

Closed interval,

$$x \in [a, b] \Rightarrow a \leq x \leq b$$

Open interval,

$$x \in (a, b) \Rightarrow a < x < b$$

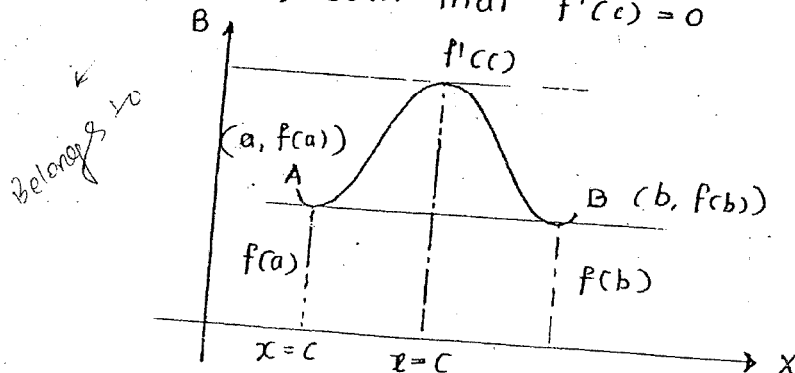
1. Rolle's mean value theorem:

Let $f(x)$ be a function defined such that $f(x)$ is

(i) $f(x)$ is continuous $[a, b]$

(ii) $f(x)$ is differentiable (a, b) exist

(iii) If $f(a) = f(b)$ the $\exists$ at least one value of $c \in (a, b)$ such that $f'(c) = 0$



$f'(c)$ slope of tangent at $x=c$ -

= slope of $\overline{AB}$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

(eqn of straight line)

$$= \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = 0$$

Find c of Rolle's mean value of function

$$f(x) = x^3 - 4x \text{ in } [-2, 2]$$

$$f'(x) = 3x^2 - 4$$

$|x|$ is continuous but
not differentiable at $x=$

Every differentiable function is continuous.

$$f(-2) = -8 + 8 = 0$$

$$f(2) = 8 - 8 = 0$$

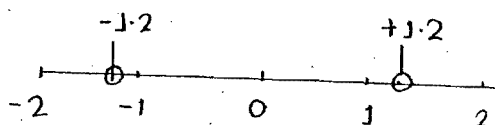
$$f'(x) = 3x^2 - 4 = 0$$

$$3x^2 = 4$$

$$x^2 = \frac{4}{3}$$

$$x = \pm \frac{2}{\sqrt{3}} = \pm 1.2$$

Given interval



— lies in the
interval

$$\text{Answer is } x = \pm \frac{2}{\sqrt{3}}$$

$$f(x) = \frac{\sin x}{e^x} \text{ in } [0, \pi]$$

a) $\frac{\pi}{6}$

b) $\frac{\pi}{4}$

c) $\frac{\pi}{3}$

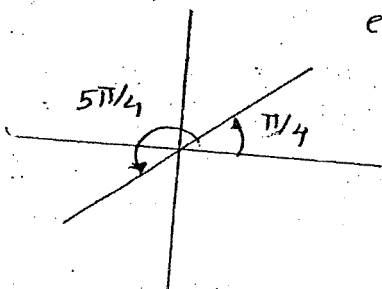
d) $\frac{\pi}{2}$

$$f'(x) = \frac{e^x \cos x - e^x \sin x}{e^{2x}}$$

$$f(0) = \frac{\sin 0}{e^0} = 0$$

$$f(\pi) = \frac{\sin \pi}{e^\pi} = \frac{0}{e^\pi} = 0$$

$$\frac{e^x \cos x - e^x \sin x}{e^{2x}} = 0$$



$$\cos x = \sin x$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\therefore x = \frac{\pi}{4}$$

$\therefore \frac{5\pi}{4}$ doesn't lie in $[0, \pi]$

$$f(x) = \frac{x^2 + ab}{x(a+b)} \quad [a, b], \quad a > 0, \quad b > 0$$

a) $\sqrt{ab}$

b) $-\sqrt{ab}$

c) $\pm \sqrt{ab}$

d) none.

$$f(a) = \frac{a^2 + ab}{a^2 + ab} = 1$$

$$f(b) = \frac{b^2 + ba}{b^2 + ab} = 1$$

$$f'(x) = \frac{1}{(a+b)} \left(\frac{x^2}{x} + \frac{ab}{x} \right)$$

$$f'(x) = \frac{1}{(a+b)} \left[1 + ab \cdot \frac{-1}{x^2} \right] = 0$$

$\therefore a > 0, b > 0$

roots will be positive

$$1 - \frac{ab}{x^2} = 0$$

$$x^2 = ab$$

$$x = \sqrt{ab}$$

Discuss applicability of Rolle's formula $f(x) = |x|$ in $[-2, 2]$

∴ function $f(x) = |x|$ is continuous function.

0 lies in between the interval $[-2, 2]$.

function of $f(x) = |x|$ is not differentiable at $x=0$

Rolle's theorem is not applicable.

2. Lagrange's Mean value theorem:

Let $f(x)$ be a function defined such that

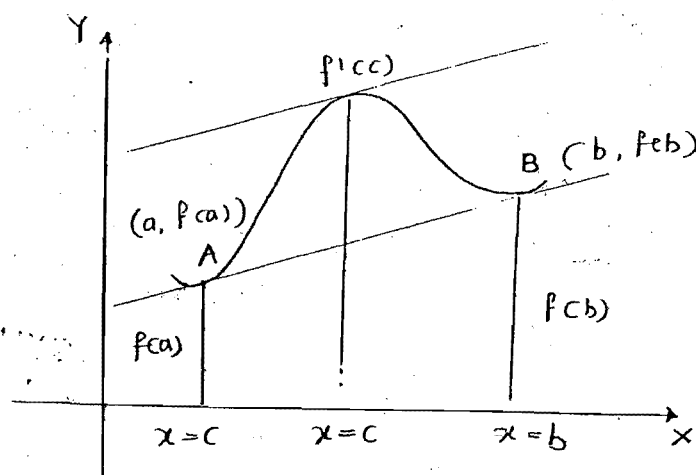
(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable in (a, b)

(iii) $f(a) \neq f(b)$

then $\exists$ at least one value $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



$f'(c)$ = slope of tgt at $x=c$

= slope of AB

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Find 'c' of Lagrange's formula for the function

$$f(x) = \log x \quad \text{in } [1, e]$$

$$(a) \frac{1}{e-1}$$

$$(b) e-1$$

$$(c) 1$$

$$(d) e$$

$$f(a) = \log 1 = 0$$

$$f(b) = \log e = 1$$

$$f'(x) = \frac{1}{x}$$

$$f'(c) = \frac{1}{c} = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{1 - 0}{e - 1}$$

$$= \frac{1}{e - 1}$$

$$c = e - 1$$

Roller

$$f(a) = f(b)$$

Lagrange's

$$f(a) \neq f(b)$$

Find 'c'

$$f(x) = x^3 - 6x^2 + 11x - 6 \quad \text{at } [0, 4]$$

$$f'(x) = 3x^2 - 12x + 11$$

$$f(a) = -6$$

$$f(b) = 64 - 96 + 44 - 6 = 6$$

$$3c^2 - 12c + 11 = \frac{6 - (-6)}{4 - 0}$$

$$3c^2 - 12c + 11 = \frac{12}{4}$$

$$3c^2 - 12c + 8 = 0$$

$$c = \frac{+12 \pm \sqrt{144 - 96}}{6}$$

$$= \frac{12 \pm 4\sqrt{3}}{6}$$

$$= 2 \pm \frac{2}{3}\sqrt{3}$$

$$= 2 \pm 2/\sqrt{3}$$

3. Cauchy's Mean value theorem:

(3)

Let $f(x)$ and $g(x)$ be two functions defined such that

(i) $f(x)$ and $g(x)$ are continuous in closed interval $[a, b]$

(ii) $f(x)$ and $g(x)$ are differential in (a, b)

(iii) $g'(x) \neq 0 \forall x \in (a, b)$ then

at least one value $c \in (a, b)$ such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Find c if $f(x)$ is e^x & $g(x)$ is e^{-x} in $[a, b]$

(a) $\frac{ab}{2}$

(b) $\frac{a+b}{2}$

(c) $\sqrt{ab}$

(d) $\frac{2ab}{a+b}$

$$f'(x) = e^x$$

$$g'(x) = -e^{-x}$$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$-e^{2c} = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}}$$

$$= -\left(\frac{e^a - e^b}{e^a - e^b}\right) \frac{1}{e^a \cdot e^b}$$

$$e^{2c} = e^{a+b}$$

$$c = \frac{a+b}{2}$$

$f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a, b]$

✓

✓

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$g'(x) = \frac{-1}{2x\sqrt{x}}$$

$$\frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}} = \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}}$$

$$\frac{-c}{f} = \frac{\sqrt{b} - \sqrt{a}}{\frac{\sqrt{a} - \sqrt{b}}{\sqrt{ab}}}$$

$$\frac{f}{c} = \sqrt{ab} \quad \in (a, b)$$

$$f(x) = \sin x, \quad g(x) = \cos x \quad \text{in } \left[0, \frac{\pi}{2}\right]$$

$$(a) \frac{\pi}{6}$$

$$(b) \frac{\pi}{4}$$

$$(c) \frac{\pi}{3}$$

$$(d) \frac{\pi}{2}$$

$$\frac{\cos c}{-\sin c} = \frac{\sin \frac{\pi}{2} - \sin 0}{\cos \frac{\pi}{2} - \cos 0}$$

$$-\cot c = \frac{1 - 0}{0 - 1}$$

$$\cot c = +1$$

$$\cot c = \cot \frac{\pi}{4}$$

$$c = \frac{\pi}{4}$$

$$f(x) = \frac{1}{x^2}, \quad g(x) = \frac{1}{x} \quad \text{in } [0, b]$$

$$\frac{\frac{-2}{x^3}}{\frac{-1}{x^2}} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}}$$

$$\frac{2}{x} = \frac{(a^2 - b^2)/a^2 b^2}{(a-b)/ab}$$

$$\frac{2}{c} = \frac{(a+b)(a-b)}{ab(a-b)}$$

$$c = \frac{2ab}{(a+b)} \in (a, b)$$

$$A.M. = \frac{a+b}{2}$$

$$G.M. = \sqrt{ab}$$

$$H.M. = \frac{2ab}{a+b}$$

4. Taylor's Mean value theorem:

Let $f(x)$ be a function defined such that

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f^{(n-1)}(x)$ is differentiable (a, b)

then, there exist at least one value $\theta \in (0, 1)$

such that

$$f(a+h) = f(a) + h \cdot f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{(n!)} f^{(n)}(a + \theta h)$$

Significance:

Any non-algebraic function can be represented as algebraic function in series form.

Taylor's formula:

$$* f(x) = f(a) + (x-a) \cdot f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$$

Maclaurin's formula, $a=0$

$$f(x) = f(0) + (x) \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

Q. Find expansion of $f(x) = e^x$ at $x=0$

$$f(x) = e^x \quad \text{at } x=0, \quad f(0) = e^0 = 1$$

$$f'(x) = e^x = 1$$

$$f''(x) = e^x = 1$$

$$f'''(x) = e^x = 1$$

$$\begin{aligned} e^x &= f(a) + (x-a) \cdot f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \\ &= 1 + x(1) + \frac{(x)^2}{2} + \frac{x^3}{6}(1) + \dots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty \end{aligned}$$

$$f(x) = \sin x \quad \text{at } x=0$$

$$\sin x = 0 + x \cdot \cos 0 + \frac{x^2}{2!} (-\sin 0) + \frac{x^3}{3!} (-\cos 0)$$

$$= x + 0 + \frac{x^3}{3!} (-1) + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$f(x) = \cos x \quad \text{at } x=0$$

$$\cos x = 1 + x \cdot (-\sin 0) + \frac{x^2}{2!} (-\cos 0) + \frac{x^3}{3!} (\sin 0)$$

$$= 1 + 0 - \frac{x^2}{2!} + 0 - \dots$$

$$f(x) = \log(1+x)$$

$$f(x) = \log(1+x) = 0 \quad \text{at } x=0$$

$$f'(x) = \frac{1}{1+x} = 1$$

$$f''(x) = \frac{-1}{(1+x)^2} = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} = 2$$

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2!} (-1) + \frac{x^3}{3!} (2) + \dots$$

$$= x - \frac{x^2}{2!} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$f(x) = \frac{x}{1+x} \quad \text{at } x=0 \quad \text{(algebraic function but not purely algebraic)}$$

$$f(x) = 1 - \frac{1}{1+x} = 0 \quad \because 1+x \mid x$$

$$f'(x) = 0 + \frac{1}{(1+x)^2} = 1 \quad \text{at } x=0$$

$$f''(x) = \frac{-2}{(1+x)^3} = -2 \quad \text{at } x=0$$

$$f'''(x) = \frac{+6}{(1+x)^4} = +6 \quad \text{at } x=0$$

$$\therefore \frac{x}{1+x} = 0 + x(1) + \frac{x^2}{2!}(-2) + \frac{x^3}{3!}(6) + \dots$$

$$= x - x^2 + x^3 - x^4 + \dots$$

or

$$x(1+x)^{-1} = x [1 - x + x^2 - x^3 + x^4 - \dots]$$

binomial expansion

$$= x - x^2 + x^3 - x^4 + \dots \infty$$

Q. Find the coefficient of x^2 in the expansion of $f(x) = \cos^2 x$ about $x=0$.

$$f(x) = \cos^2 x$$

$$= \cos x \cdot \cos x$$

Expansion of $f(x)$ is

$$\cos^2 x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)$$

$$\text{coeff of } x^2 = \frac{-1}{2!} - \frac{1}{2!}$$

$$= \frac{-2}{2!}$$

$$= -1$$

or

Maclaurine's formula:

$$f(x) = f(0) + x \cdot f'(0) + \boxed{\frac{x^2}{2!} f''(0)} + \dots$$

$$f(x) = \cos^2 x$$

$$f'(x) = 2 \cos x (-\sin x)$$

$$= -2 \sin x \cdot \cos x$$

$$= -\sin 2x$$

$$f''(x) = -\cos 2x \times 2 = -2 \quad \text{at } x=0$$

$$f'''(x) = 2 \times 2 \sin 2x$$

$$\text{coeff of } x^2 = \frac{f''(0)}{2!} = \frac{-2}{2} = -1$$

$f(x) = (x-2)^4$ in the expansion of $f(x) = e^x$ about $x=2$

(a) $\frac{1}{2!}$

(b) $\frac{1}{4!}$

(c) $\frac{e^4}{4!}$

(d) $\frac{e^2}{4!}$

$$\begin{aligned} \therefore \text{coefficient of } (x-2)^4 &= \frac{f^{(4)}(2)}{4!} \\ &= \frac{e^2}{4!} \end{aligned}$$

Find expansion of $f(x) = x^2$ about $x=1$

(a) $1-x+x^2$

(b) $1+x^2$

(c) x^2

(d) $1-x + \frac{x^2}{2!} - \frac{x^3}{3!}$

The given function is pure algebraic. thus there will be no expansion.

$$\therefore f(x) = x^2 \text{ at } x=1$$

$$\text{Expansions} = x^2 \text{ (itself)}$$

Find the expansion of $f(x) = \sin x$ at $x=\pi$

By Taylor's formula,

$$f(x) = \sin x = 0 \text{ at } x=\pi$$

$$f'(x) = \cos x = -1$$

$$f''(x) = -\sin x = 0$$

$$f'''(x) = -\cos x = 1$$

$$\begin{aligned} \sin x &= 0 + (x-\pi)(-1) + \frac{(x-\pi)^2}{2!}(0) + \frac{(x-\pi)^3}{3!}(1) + \dots \\ &= -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} + \frac{(x-\pi)^7}{7!} + \dots \end{aligned}$$

* Partial and total derivative →

Tuesday
24th September 2013

Q.2

Ordinary derivative

$$y = f(x)$$

$$z = g(y)$$

Partial derivative

$$z = f(x, y)$$

$$u = f(x, y, z)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

$$z = x^2 - xy + y^2$$

$$\frac{\partial z}{\partial x} = 2x - y \Rightarrow \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} (2x - y) = -1$$

$$\frac{\partial^2 z}{\partial x \partial y} = -1$$

$$\frac{\partial^2 z}{\partial x \partial y} = -1$$

$$\frac{\partial z}{\partial y} = -x + 2y$$

$$\frac{\partial z}{\partial y \partial x} = -1$$

Homogenous function:

A function $f(x, y)$ is said to be homogenous of degree n in x and y if

$$f(kx, ky) = k^n \cdot f(x, y)$$

e.g. $f(x, y) = x^2 - xy + y^2$

$$\begin{aligned} f(kx, ky) &= k^2 x^2 - k^2 xy + k^2 y^2 \\ &= k^2 (x^2 - xy + y^2) \end{aligned}$$

$$f(kx, ky) = k^2 f(x, y)$$

Note:

(i) The product of two homogenous functions is again a homogenous function.

$$f(x, y) = (x^3 + y^3)(x^2 - y^2)$$

$$n = 3 + 2 = 5$$

① If function is in rational form, if both numerator and the denominator are homogenous functions then given function is also homogenous.

e.g. $f(x, y) = \frac{x^3 + y^3}{x^2 - y^2}$

$$\text{degree, } n = 3 - 2 = 1$$

Euler's theorem:

If z is a homogenous function of degree n in x and y then

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n \cdot z$$

Note:

This formula is applicable directly if z is algebraic function.

e.g. If $z = (x^2 + y^2)^{1/3}$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \frac{2}{3} z \quad \therefore n = 2 \times \frac{1}{3} = \frac{2}{3}$$

If $z = \frac{x'y'}{x' + y'}$

$$x \cdot z_x + y \cdot z_y = 1 \cdot z$$

$$n = 2 - 1 = 1$$

If $z = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n z$$

Note:

If z is not algebraic and let $\phi(z)$ is algebraic and homogenous of degree n in x and y the 34

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \frac{n \cdot \phi(z)}{\phi'(z)}$$

e.g. If $z = \log_e(x^2 + y^2)$

$$e^z = (x^2 + y^2)$$

$$n = 2$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \frac{2 \cdot e^z}{e^z}$$

$$\phi(z) = e^z$$

$$= 2$$

1. $z = \sin^{-1}\left(\frac{x^2 + y^2}{x - y}\right)$

$$\sin z = \frac{x^2 + y^2}{x - y}$$

$$n = (2 - 1) = 1$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 1 \cdot \frac{\sin z}{\cos z}$$

$$= \tan z$$

2. $z = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$

$$\tan z = \frac{x^3 + y^3}{x - y}$$

$$n = (3 - 1) = 2$$

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 2 \cdot \frac{\tan z}{\sec^2 z}$$

$$= 2 \cdot \frac{\sin z}{\cos z} \cdot \frac{\cos^2 z}{1}$$

$$= 2 \sin z$$

$$= \sin 2z$$

3. $z = x^2 \cdot \sin^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Though it contains trigonometric functions for $\left(\frac{1}{x}\right)$ & $\left(\frac{x}{y}\right)$ the degree is zero $(1-1)=0$ and $(1-1)=0$.

Thus we consider there exist no inverse function.

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$$

4. $z = (x^3 + y^3) \cdot e^{-x/y}$

$$n = 3$$

$$\therefore \text{for } e^{-x/y}, n = +1 - (1) = 0$$

Application of Euler's theorem:

If z is homogenous function of degree n in x, y then,

$$x^2 \cdot \frac{\partial^2 z}{\partial x^2} + 2xy \cdot \frac{\partial^2 z}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

This formula is applicable only when z is applicable

e.g. If $z = (x^2 + y^2)^{1/3}$

$$\begin{aligned} x^2 \cdot \frac{\partial^2 z}{\partial x^2} + 2xy \cdot \frac{\partial^2 z}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 z}{\partial y^2} &= \frac{2}{3} \left(\frac{2}{3} - 1\right) \cdot z \\ &= \frac{2}{3} \left(-\frac{1}{3}\right) \cdot z \\ &= -\frac{2}{9} z \end{aligned}$$

If $z = \frac{xy}{x+y}$

$$\therefore n = (1+1) - 1 = 1$$

$$\begin{aligned} x^2 \cdot \frac{\partial^2 z}{\partial x^2} + 2xy \cdot \frac{\partial^2 z}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 z}{\partial y^2} &= 1(1-1) \cdot z \\ &= 0 \end{aligned}$$

If $z = \tan^{-1} \left(\frac{x^2 + y^2}{x - y} \right)$

$$\tan z = \frac{x^2 + y^2}{x - y}$$

$$\therefore x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = z \cdot \frac{\tan z}{\tan^2 z} = \sin 2z$$

$$\therefore x \cdot \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} (1) + y \cdot \frac{\partial^2 z}{\partial x \partial y} = 2 \cos 2z \cdot \frac{\partial z}{\partial x}$$

$$\therefore x \cdot \frac{\partial^2 z}{\partial x^2} + y \cdot \frac{\partial^2 z}{\partial x \partial y} = (-1 + 2 \cos 2z) \cdot \frac{\partial z}{\partial x}$$

$$\therefore x^2 \cdot \frac{\partial^2 z}{\partial x^2} + xy \cdot \frac{\partial^2 z}{\partial x \partial y} = (2 \cos 2z - 1) \cdot x \cdot \frac{\partial z}{\partial x} \quad \text{--- a}$$

Similarly

$$y^2 \cdot \frac{\partial^2 z}{\partial y^2} + xy \cdot \frac{\partial^2 z}{\partial x \partial y} = (2 \cos 2z - 1) \cdot y \cdot \frac{\partial z}{\partial y} \quad \text{--- b}$$

Adding --- a) & --- b)

$$\therefore x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \cdot \frac{\partial^2 z}{\partial x \partial y} + y^2 \cdot \frac{\partial^2 z}{\partial y^2} = (2 \cos 2z - 1) \cdot \sin 2z = \sin 4z - \sin 2z$$

* Explicit function:

Any function which we can express in the form of either $y = f(x)$ or $x = g(y)$ etc. is called an explicit function.

e.g. if

$$y = ax^2 + bx + c$$

$$\frac{dy}{dx} = 2ax + b$$

$$\frac{d^2y}{dx^2} = 2a$$

$$\frac{d^3y}{dx^3} = 0$$

Implicit function:

Any function which is not explicit is said to be an implicit function.

e.g. $ax^2 + 2hxy + by^2 = 0$

$$\therefore a \cdot 2x + 2h \left[x \cdot \frac{dy}{dx} + y(1) \right] + 2by \cdot \frac{dy}{dx} = 0$$

$$(ah + by) \cdot \frac{dy}{dx} = -ax + hy$$

$$\frac{dy}{dx} = \frac{-ax + hy}{(ah + by)}$$

$$\frac{d^2y}{dx^2} = \dots \text{ (complicated \& lengthy)}$$

Partial derivatives are easier & simpler than ordinary derivatives because we treat y as constant while differentiating given function w.r.t. x & x as constant while differentiating given function w.r.t. y .

Rememo

$$z = f(x, y)$$

$$p = \frac{\partial z}{\partial x}$$

$$q = \frac{\partial z}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-[q^2r - 2pq s + p^2t]}{q^3}$$

(for implicit function)

~~***~~

$$ax^2 + 2hxy + by^2 = 1$$

(particular formula)

$$\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}$$

$$1. \quad 2x^2 + 4xy + 3y^2 = 1$$

$$a=2, \quad b=3, \quad h=\frac{4}{2}=2$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{h^2 - ab}{(hx + by)^3} \\ &= \frac{4 - 6}{(2x + 3y)^3} \\ &= \frac{-2}{(2x + 3y)^3} \end{aligned}$$

$$2x^2 + 4xy + 3y^2 = 1$$

$$ax^2 + 2hxy + by^2 = 1$$

$$a=2, \quad \cancel{2hxy} = \cancel{4xy}$$

$$\begin{aligned} h &= 2 \\ b &= 3 \end{aligned}$$

$$2. \quad x^2 + xy + y^2 = 1$$

$$a=1, \quad b=1, \quad h=\frac{1}{2}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{h^2 - ab}{(-\frac{1}{2}x + y)^3} \\ &= \frac{-3/4}{(\frac{x+y}{2})^3} \\ &= \frac{-2 \times 3}{(x+y)^3} \\ &= \frac{-6}{(x+y)^3} \end{aligned}$$

If given function is in the form $f = g = 0 \quad c > 0$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = \text{const}$$

then

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$$

— (standard formula)

(Applicable for all forms above)

$$2x^2 + 4xy + 3y^2 + 2x + 4y + 1 = 0$$

$$a=2, b=3, c=1, h=\frac{4}{2}=2, g=\frac{2}{2}=1, f=\frac{1}{2}=2$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3} \\ &= \frac{6 + 8 - 8 - 3 - 4}{(2x + 3y + 2)^3} \\ &= \frac{-1}{(2x + 3y + 2)^3}\end{aligned}$$

1. $y^2 - 5x + 4x^2 = 8$ find $\frac{d^2y}{dx^2} = ?$

$$4x^2 - 5x - y^2 - 8 = 0$$

$$a=4, b=-1, h=0, g=\left(-\frac{5}{2}\right), f=0, c=-8$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{32 + 0 - 0 + (-1)\frac{25}{4} - 0}{(0 - 1 \cdot y + 0)^3} \\ &= \frac{32 - \frac{25}{4}}{-y^3} \\ &= \frac{-153}{4y^3}\end{aligned}$$

2. $y^2 - 5x + 4x^2 = 8$ find $\frac{d^2y}{dx^2}$

$$y^2 - 5x + 4x^2 = 8 \quad (\text{explicit function})$$

$$a=4, b=1, g=\left(-\frac{5}{2}\right), c=0, h=0, f=0$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{0 + 0 - 0 - \frac{25}{4} - 0}{(0 + y + 0)^3} \\ &= \frac{-25}{4y^3}\end{aligned}$$

Composite function →

(34)

Any function in function is said to be composite function.

If z is function in x and y and x, y are function in t then z is called a composite function in t .

Total derivative of composite function:

If $z = f(x, y)$, $x = g(t)$ & $y = h(t)$ then

total derivative of z w.r.t. t is denoted by $\frac{dz}{dt}$ and is given by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

If $u = f(x, y, z)$ & (x, y, z) are function of t

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

If $z = e^x \cdot \sin y$, $x = \log t$, $y = t^2$

$$\frac{\partial z}{\partial x} = e^x \cdot \sin y$$

$$\frac{dx}{dt} = \frac{1}{t}$$

$$\frac{\partial z}{\partial y} = e^x \cdot \cos y$$

$$\frac{dy}{dt} = 2t$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (e^x \cdot \sin y) \cdot \frac{1}{t} + (e^x \cdot \cos y) \cdot 2t$$

$$= \frac{e^x}{t} [\sin y + 2t^2 \cos y]$$

If $u = x^2 + y^2 + z^2$

& $x = e^{2t}, y = e^{2t} \cos 3t, z = e^{2t} \sin 3t$

$$\begin{aligned} u &= (e^{2t})^2 + (e^{2t} \cos 3t)^2 + (e^{2t} \sin 3t)^2 \\ &= e^{4t} (1 + \cos^2 3t + \sin^2 3t) \\ &= e^{4t} (1 + 1) \\ &= 2e^{4t} \end{aligned}$$

$$\frac{du}{dt} = 8t \cdot e^{4t}$$

If $u = x^2 - y^2, x = e^t \cos t, y = e^t \sin t$

$$u = (e^t \cos t)^2 - (e^t \sin t)^2$$

$$u = e^{2t} [\cos^2 t - \sin^2 t]$$

$$\frac{du}{dt} = \frac{d}{dt} e^{2t} (\cos 2t)$$

$$= e^{2t} \cdot \frac{d}{dt} \cos 2t + \cos 2t \cdot \frac{d}{dt} e^{2t}$$

$$= -2 \sin 2t \cdot e^{2t} + 2 \cdot e^{2t} \cdot \cos 2t$$

$$\begin{aligned} \left(\frac{du}{dt} \right)_{t=0} &= 0 + 2 \\ &= 2 \end{aligned}$$

If $u = x^3 \cdot y \cdot e^z$

$x = t, y = t^2, z = \log t$

$$u = t^3 \cdot t^2 \cdot e^{\log t}$$

$$= t^5 \cdot t$$

$$= t^6$$

$$\begin{aligned} \left(\frac{du}{dt} \right)_{t=2} &= 6 \cdot t^5 \\ &= 6 \times (2)^5 = 6 \times 32 = 192 \end{aligned}$$

If $u = e^{xyz}$, $\frac{\partial^2 u}{\partial x \cdot \partial y \cdot \partial z}$

(28)

(a) $e^{xyz} \cdot (1 + z + y + z)$

(b) $e^{xyz} (x^2 + y^2 + z^2)$

(c) $e^{xyz} (1 + 3xyz + x^2 y^2 z^2)$

(d) None

$u = e^{xyz}$

$\frac{d}{dx} (e^{ax}) = a \cdot e^{ax}$

$\frac{\partial u}{\partial x} = yz \cdot e^{xyz}$

$\frac{\partial u}{\partial x \cdot \partial y} = z \left[y \cdot xz \cdot e^{xyz} + e^{xyz} \cdot (1) \right]$

$\frac{\partial u}{\partial x \cdot \partial y} = xyz^2 \cdot e^{xyz} + z \cdot e^{xyz}$

$\frac{\partial^2 u}{\partial x \cdot \partial y \cdot \partial z} = xy \left[z^2 \cdot xy \cdot e^{xyz} + e^{xyz} \cdot 2z \right] + \left[z \cdot e^{xyz} \cdot xy + e^{xyz} \right]$
 $= x^2 y^2 z^2 \cdot e^{xyz} + 2xyz \cdot e^{xyz} + z \cdot xy \cdot e^{xyz} + e^{xyz}$
 $= e^{xyz} \left[x^2 y^2 z^2 + 3xyz + 1 \right]$

If $f(x) = y^x$ find $\frac{\partial^2 f}{\partial x \cdot \partial y}$ at $x=2, y=1$

$f = y^x$

$\frac{\partial f}{\partial y} = x \cdot y^{x-1}$

$= x \cdot (y)^{x-1}$

at $y=1$

$\frac{\partial^2 f}{\partial x \cdot \partial y} = 1$

If $z = f(x+ay) + \phi(x-ay)$ then which of the following is true.

(a) $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$

(b) $\frac{\partial^2 z}{\partial x^2} = -a^2 \frac{\partial^2 z}{\partial y^2}$

(c) $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

(d) $\frac{\partial^2 z}{\partial y^2} = -a^2 \frac{\partial^2 z}{\partial x^2}$

$$z = f(x+ay) + \phi(x-ay)$$

$$\frac{\partial z}{\partial x} = f'(x+ay) \cdot (1) + \phi'(x-ay) \cdot (1)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) \cdot (1) + \phi''(x-ay) \cdot (1)$$

$$\frac{\partial z}{\partial y} = f'(x+ay) \cdot a + \phi'(x-ay) \cdot (-a)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x+ay) \cdot a^2 + \phi''(x-ay) \cdot a^2$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

If $x = r \cos \theta$

$y = r \sin \theta$

$$\frac{\partial \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial^2 y} = ?$$

$$\frac{\partial^2 \theta}{\partial x \cdot \partial y} = ?$$

$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta}$$

$$\frac{y}{x} = \tan \theta$$

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left(\frac{y}{x} \right)^2} \left(\frac{-y}{x^2} \right)$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2}$$

(39) ——— (1)

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{-1}{(x^2+y^2)^2} \cdot (-y) \cdot 2x$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

similarly

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

from — (1)

$$\frac{\partial \theta}{\partial x \cdot \partial y} = \frac{(x^2+y^2) \cdot (-1) - (-y) \cdot (2x)}{(x^2+y^2)^2}$$

$$= \frac{-x^2 - y^2 + 2y^2}{(x^2+y^2)^2}$$

$$= \frac{-(x^2 - y^2)}{(x^2+y^2)^2}$$

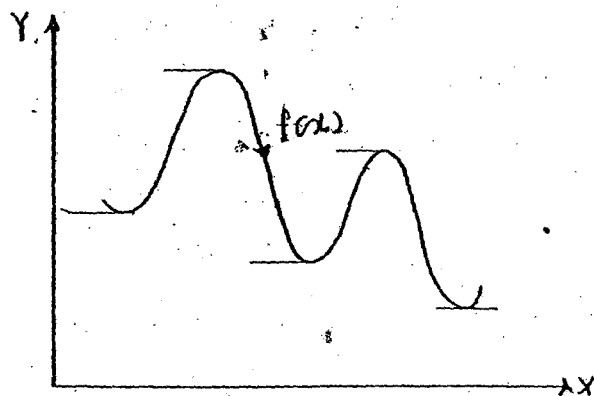
$$= \frac{-(r^2 \cos^2 \theta - r^2 \sin^2 \theta)}{(r^2 \cos^2 \theta + r^2 \sin^2 \theta)^2}$$

$$= \frac{-r^2 (\cos 2\theta)}{(r^2 (1)^2)}$$

$$= \frac{-r^2 \cos 2\theta}{r^4}$$

$$= \frac{-\cos 2\theta}{r^2}$$

Maxima and Minima :



Maximum value :

If a continuous function $f(x)$ increases to a certain value and then decreases, that value is called maximum value of a function.

Minimum value :

If a continuous function $f(x)$ decreases to a certain value and then increases that value is called minimum value of the function.

- (i) The maxima or minima occurs alternatively.
- (ii) A function can have several maximum and several minimum values.
- (iii) The minimum value may be greater than maximum value.
- (iv) The least minimum value is called global minimum or universal minimum and highest maximum value is called a global maximum or universal maximum.

For a function $f(x)$, for maxima or minima,

$$f'(x) = 0$$

$$x = \alpha, \beta, \gamma, \dots$$

For $x = \alpha$, if $f''(\alpha) > 0$ the $f(x)$ is minimum at α .

$$\text{Minimum value} = f(\alpha)$$

(40)

For $x = \beta$, if $f''(\beta) < 0$, the $f(x)$ is maximum at β .

$$\text{Maximum value} = f(\beta)$$

For $x = \gamma$, if $f''(\gamma) = 0$, the $f(x)$ is stationary.

if $f'''(\gamma) \neq 0$, then γ is said to be inflection point

1. $f(x) = x^x$

$$f'(x) = x^x (1 + \log x)$$

$$\begin{aligned} f''(x) &= x^x \left(-\frac{1}{x}\right) + (1 + \log x) \cdot x^x \cdot (1 + \log x) \\ &= \frac{x^x}{x} + (1 + \log x)^2 \cdot x^x \end{aligned}$$

For maxima or minima,

$$f'(x) = x^x (1 + \log x)$$

$$f'(x) = 0$$

$$x^x (1 + \log x) = 0$$

$$x^x \neq 0 \quad \text{for any value of } x$$

$$1 + \log x = 0$$

$$\log x = -1$$

$$x = e^{-1} = \frac{1}{e}$$

$$f''\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e-1} + \left(1 + \log_e \frac{1}{e}\right)^2$$

$$2^2 = 4 > 0$$

$$2^{-2} = \frac{1}{4} > 0$$

$$= \left(\frac{1}{e}\right)^{1/e-1} + 0$$

$$> 0$$

+ve $\rightarrow$ min

-ve $\rightarrow$ max

$f(x)$ is minimum at $x = \frac{1}{e}$

$$\begin{aligned} \text{Minimum value} &= f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e} \\ &= (e^{-1})^{1/e} \\ &= e^{-1/e} \end{aligned}$$

$f(x) = x^{1/x}$ has its maxima at $x = e$.

Max. value $= e^{1/e}$

~~Short trick~~

function

Minimum value

Maximum value

at

x^2

$e^{-1/e}$

-

$x = 1/e$

$x^{1/x}$

-

$e^{1/e}$

$x = e$

$$f(x) = \frac{\log x}{x}$$

$$f'(x) = \frac{x \left(\frac{1}{x} \right) + \log x}{x^2}$$

$$= \frac{1 + \log x}{x^2}$$

$$f''(x) = \frac{x^2 \left(\frac{1}{x} \right) + 2x \cdot (1 + \log x)}{x^4}$$

$$= \frac{x (1 + 2(1 + \log x))}{x^4}$$

$$= \frac{1 + 2(1 + \log x)}{x^3}$$

For max. or minima,

$$f'(x) = 0$$

$$\frac{1 + \log x}{x^2} = 0$$

$$\log x = -1$$

$$x = e^{-1}$$

$$f''(e^{-1}) = \frac{1 - 2(1 - \log e)}{e^3}$$

$$= \frac{-1}{e^3} < 0$$

$$\text{Max. value} = \frac{\log e}{e} = \frac{1}{e}$$

$$f(x) = 3x^2 - 6x$$

$$f'(x) = 6x - 6$$

$$f''(x) = 6$$

$$6x - 6 = 0$$

$$x = 1$$

$$f(1) = 3 - 6$$

$$= -3 < 0$$

Maximum value = -3

$$f(x) = x^3 + \frac{3}{x^2}$$

$$f'(x) = 3x^2 - \frac{3}{x^2} = 0$$

$$3x^2 = \frac{3}{x^2}$$

$$x^4 = 1$$

$$(x^2 - 1)(x^2 + 1) = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

$$f''(x) = 6x + \frac{6}{x^3}$$

$$\text{At } x = +1$$

$$f''(1) = 6(1) + \frac{6}{(1)}$$

$$= 12 > 0$$

Minimum at $x = 1$

$$\text{At } x = -1$$

$$f''(-1) = 6(-1) + \frac{6}{(-1)}$$

$$= -6 - 6$$

$$= -12 < 0$$

Maximum value at $x = -1$

$$f(x) = x^2 \cdot e^{-x}$$

$$f'(x) = x^2 \cdot e^{-x} \cdot (-1) + e^{-x} \cdot 2x$$

$$f''(x) = \frac{d}{dx} [e^{-x} (-x^2 + 2x)]$$

$$= 2 \cdot e^{-x} - 2 \cdot x \cdot e^{-x} - 2x \cdot e^{-x} + x^2 \cdot e^{-x}$$

$$= e^{-x} (2 - 4x + x^2)$$

$$f'(x) = 0$$

$$e^{-x} (2x - x^2) = 0$$

$$x(2 - x) = 0$$

$$x = 0, \quad \text{or} \quad x = 2$$

$$\text{At } x = 0$$

$$f''(0) = e^{-0} (2 - 0 + 0)$$

$$= 2 > 0$$

Minimum at $x = 0$

$$\text{At } x = 2$$

$$f''(2) = e^{-2} (2 - 8 + 4)$$

$$= \frac{-2}{e^2}$$

$$f(x) = y = x^{5/2}$$

$$\frac{dy}{dx} = \frac{5}{2} x^{3/2}$$

$$\frac{d^2y}{dx^2} = \frac{5}{2} \times \frac{3}{2} \cdot x^{1/2}$$

$$\frac{d^3y}{dx^3} = \frac{15}{8} \frac{1}{\sqrt{x}}$$

$$\text{At } x = 0,$$

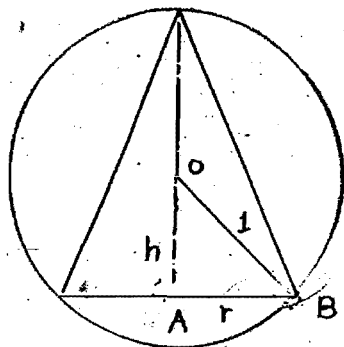
$$\frac{dy}{dx} = 0 \Rightarrow x = 0$$

$$\text{At } x = 0,$$

$$\frac{d^3y}{dx^3} = \frac{15}{8} \frac{1}{\sqrt{0}} = \infty \neq 0$$

inflection point at $(0, 0)$

Find the height of cone of maximum volume that can be inscribed in the sphere of radius 1 unit. — (42)



$$OA^2 + AB^2 = OB^2$$

$$h^2 + r^2 = 1^2$$

$$r^2 = 1^2 - h^2$$

volume of cone,

$$V = \frac{1}{3} \pi r^2 h$$

$$= \frac{\pi}{3} (1 - h^2)(h + 1)$$

$$= \frac{\pi}{3} (1 + h - h^2 - h^3)$$

height of cone
= $h + 1$

$$\therefore \frac{dV}{dh} = \frac{\pi}{3} (0 + 1 - 2h - 3h^2)$$

$$\therefore \frac{d^2V}{dh^2} = \frac{\pi}{3} (-2 - 6h)$$

for maxima,

$$\therefore \frac{dV}{dh} = 0$$

$$\frac{\pi}{3} (1 - 2h - 3h^2) = 0$$

$$3h^2 + 2h - 1 = 0$$

$$(3h - 1)(h + 1) = 0$$

$$h = \frac{1}{3}$$

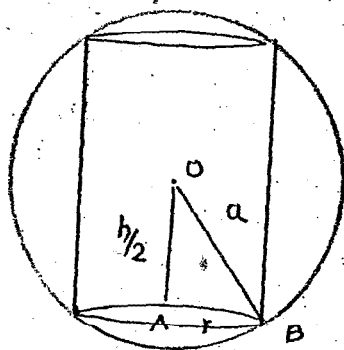
(height cannot be -ve)

$$\text{height of the cone} = 1 + h$$

$$= 1 + \frac{1}{3}$$

$$= \frac{4}{3}$$

Q Find the height of cylinder of max. volume that can be inscribed in the sphere of radius a units.



$$OA^2 + OB^2 = a^2$$

$$\left(\frac{h}{2}\right)^2 + r^2 = a^2$$

$$r^2 = a^2 - \frac{h^2}{4}$$

volume of cylinder,

$$= \pi r^2 \cdot h$$

$$= \pi \left(a^2 - \frac{h^2}{4}\right) \cdot h$$

$$= \pi \left(a^2 h - \frac{h^3}{4}\right)$$

$$\frac{dv}{dh} = \pi \left(a^2 - \frac{3h^2}{4}\right)$$

for maximum volume

$$\frac{dv}{dh} = 0$$

$$a^2 - \frac{3h^2}{4} = 0$$

$$a = \frac{\sqrt{3}h}{2}$$

$$h = \frac{2a}{\sqrt{3}} *$$

For $z = f(x, y)$

$$p = \frac{\partial z}{\partial x}$$

$$q = \frac{\partial z}{\partial y}$$

$$s = \frac{\partial^2 z}{\partial x \partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

(13)

For a maxima or minima, Let $p=0$ and $q=0$ since p and q are functions in x and y , so we have two equations in x and y . Solve these equations and get relation between x and y .

e.g. $(x=y, x=-y, x=2y \text{ etc.})$

Put this relation either in $p=0$ or in $q=0$, then the equation transformed to either purely in x or in y . Solve the equation and get the roots

$$x = x_0, x_1, x_2 \dots$$

$$y = y_0, y_1, y_2 \dots$$

$(x_0, y_0), (x_1, y_1), (x_2, y_2) \dots$ are called stationary points or critical points (points at which maxima, minima can be decided)

Take point (x_0, y_0) and find values of r, s and t .

Case I → If $rt - s^2 > 0$ and $r > 0$, then function attains its minimum.

$$\text{Minimum value} = f(x_0, y_0)$$

Case II → If $rt - s^2 > 0$ and $r < 0$ then function attains its maximum.

$$\text{Maximum value} = f(x_1, y_1)$$

Case III → If $rt - s^2 < 0$ then the function has no extreme value (neither minima nor maxima)

Case IV → If $rt - s^2 = 0$, the case is doubtful and needs further investigation.

Q. The function $f(x) = x^2 + y^2 + 6x + 12$ has

- a) maximum at $(3, 0)$ cb) min at $(-3, 0)$
 co) no extreme at $(-3, 0)$ cd) none

$$p = \frac{\partial f}{\partial x} = 2x + 6$$

$$q = \frac{\partial f}{\partial y} = 2y$$

$$r = \frac{\partial^2 f}{\partial x^2} = 2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

$$rt - s^2 = (2 \times 2) - 0 = 4 > 0$$

$$r = 2 > 0$$

$f(x)$ has minimum value at $(-3, 0)$

Q. $f(x, y) = 1 - x^2 - y^2$ has

- a) minimum at $(0, 0)$ cb) max at $(0, 0)$
 co) no extreme at $(0, 0)$ cd) none

$$p = \frac{\partial f}{\partial x} = -2x$$

$$q = \frac{\partial f}{\partial y} = -2y$$

$$r = \frac{\partial^2 f}{\partial x^2} = -2$$

$$s = \frac{\partial^2 f}{\partial y \partial x} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = -2$$

$$rt - s^2 = (-2)(-2) - 0$$

$$= 4 > 0$$

$$r = -2 < 0$$

$f(x)$ has max. value at $(0, 0)$

Q. $f(x, y) = x^3 + y^3 - 3xy$

- a) Min at $(1, -1)$ cb) Max at $(1, -1)$
 co) Min at $(1, 1)$ cd) Max at $(1, 1)$

$$p = \frac{\partial f}{\partial x} = 3x^2 - 3y$$

$$q = \frac{\partial f}{\partial y} = 3y^2 - 3x$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = -3$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6y - 6$$

(44)

for (1, -1),
 $rt - s^2 =$
 $p \neq 0, \quad q \neq 0$

for (1, 1)
 $p = 3(1)^2 - 3(1) = 0$
 $q = 3(1)^2 - 3(1) = 0$

$$rt - s^2 = 6 \times 6 - 9 > 0$$

$$r = 6 > 0$$

Function has minima at (1, 1)

Q. $f(x, y) = x^2 + y^2 + xy + x - 4y + 5$ has

(a) Min. at (-2, 3)

(b) Max. at (-2, 3)

(c) Min at (2, -3)

(d) Max. at (2, -3)

$$p = \frac{\partial f}{\partial x} = 2x + y + 1$$

$$q = \frac{\partial f}{\partial y} = 2y + x - 4$$

$$r = \frac{\partial^2 f}{\partial x^2} = 2$$

$$s = \frac{\partial^2 f}{\partial y \partial x} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

At (-2, 3),

$$p = 2(-2) + 3 + 1$$

$$= 0$$

$$q = 2(3) + (-2) - 4$$

$$= 0$$

$$(rt - s^2) = 4 - 1 > 0$$

$$r > 0$$

Minimum at (-2, 3)

Q. 1049) $f(x, y) = 4x + 6y + 8 - 8x - 4y$ has (value = ?)

(a) Max at $8/3$

(b) Min at $8/3$

(c) Max at $10/3$

(d) Max - $10/3$

$$p = \frac{\partial f}{\partial x} = 8x - 8$$

$$q = \frac{\partial f}{\partial y} = 12y - 4$$

$$r = \frac{\partial^2 f}{\partial x^2} = 8$$

$$s = \frac{\partial^2 f}{\partial y \partial x} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = 12$$

$$p = 0$$

$$8x - 8 = 0$$

$$x = 1$$

$$q = 0$$

$$12y - 4 = 0$$

$$y = \frac{1}{3}$$

At $(x=1, y=\frac{1}{3})$

$$f(1, \frac{1}{3}) = 4(1)^2 + 6(\frac{1}{3})^2 + 8 - 8(1) - 4(\frac{1}{3})$$

$$= \frac{10}{3}$$

$$rt - s^2 = 96 > 0$$

$$r = 8 > 0$$

$$\text{Minimum value} = \frac{10}{3}$$

Q. $f(x, y) = \sin x + \sin y + \sin(x+y)$ has

(a) Max $\frac{\sqrt{3}}{2}$

(b) Min $\frac{\sqrt{3}}{2}$

(c) Max $\frac{3\sqrt{3}}{2}$

(d) Min $\frac{3\sqrt{3}}{2}$

$$p = \frac{\partial f}{\partial x} = \cos x + \cos(x+y)$$

$$q = \frac{\partial f}{\partial y} = \cos y + \cos(x+y)$$

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$$

$$s = \frac{\partial^2 f}{\partial y \partial x} = -\sin(x+y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$$

$$p = 0$$

$$\cos x + \cos(x+y) = 0$$

$$q = 0$$

$$\cos y + \cos(x+y) = 0$$

Solving — (i) & — (ii)

$$\cos x - \cos y = 0$$

$$x = y$$

— (iii)

from — (i)

$$\cos x + \cos 2x = 0$$

$$\cos x \times 2 = -\cos 2x$$

$$\cos x \times 2 = \cos(\pi - x)$$

$$2x = \pi - x$$

$$x = \frac{\pi}{3}$$

$$y = \frac{\pi}{3}$$

$$r = -\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right) = -\sqrt{3}$$

$$s = -\sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$t = -\sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right) = -\sqrt{3}$$

$$rt - s^2 = 3 - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= 3 - \frac{3}{4} = > 0$$

$$r = -\sqrt{3} < 0$$

Max: at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$\text{Max. value} = \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \cos\left(90^\circ + \frac{\pi}{6}\right)$$

$$= \frac{3\sqrt{3}}{2}$$

Definite and Improper Integrals

$$\int_a^b f(x) \cdot dx = \int_a^c f(x) \cdot dx + \int_c^b f(x) \cdot dx.$$

$$\int_a^b f(x) \cdot dx = \int_a^b (a+b-x) \cdot dx$$

$$\int_a^b \frac{f(x)}{f(x) + f(a+b-x)} \cdot dx = \frac{b-a}{2}$$

$$\int_a^b f(x) \cdot dx = \int_0^a f(a-x) \cdot dx.$$

$$\int_{-a}^a f(x) \cdot dx = 2 \int_0^a f(x) \cdot dx, \text{ if } f(x) \text{ is even, i.e. } f(-x) = f(x)$$

$$= 0 \text{ if } f(x) \text{ is odd i.e. } f(-x) = -f(x)$$

$$\int_0^{2a} f(x) \cdot dx = 2 \int_0^a f(x) \cdot dx, \text{ if } f(2a-x) = f(x)$$

$$= 0 \text{ if } f(2a-x) = -f(x).$$

$$\int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x) + f(\cos x)} \cdot dx = \int_0^{\pi/2} \frac{f(\tan x)}{f(\tan x) + f(\cot x)} \cdot dx = \int_0^{\pi/2} \frac{f(\sec x)}{f(\sec x) + f(\csc x)} \cdot dx$$

$$\int_0^{\pi/2} \sin^n x \cdot dx = \int_0^{\pi/2} \cos^n x \cdot dx = \frac{(n-1)(n-3)(n-5) \dots \times k}{n(n-2)(n-4) \dots}$$

$$k=1 \text{ if } n \text{ is odd.}$$

$$k=\pi/2 \text{ if } n \text{ is even.}$$

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x \cdot dx = \frac{(m-1)(m-3)\dots(m-1)(n-3)\dots}{(m+n)(m+n-2)\dots(m+n-4)} \dots k. \quad (41)$$

$k = 1$ if either m or n or both are odd.

$k = \frac{\pi}{2}$ if both m and n are even.

$$\int x^n \cdot dx = \frac{x^{n+1}}{n+1}$$

$$\int \sinh x \cdot dx = \cosh x$$

$$\int \frac{1}{x} \cdot dx = \log x$$

$$\int \cosh x \cdot dx = \sinh x.$$

$$\int \frac{1}{x^2} \cdot dx = \frac{-1}{x}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1} \left(\frac{x}{a} \right)$$

$$\int \frac{1}{x^3} \cdot dx = \frac{-1}{2x^2}$$

$$\int \frac{1}{\sqrt{a^2 + x^2}} \cdot dx = \sin^{-1} h \cdot \left(\frac{x}{a} \right)$$

$$\int e^{ax} \cdot dx = \frac{e^{ax}}{a}$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \cdot dx = \cosh^{-1} \left(\frac{x}{a} \right)$$

$$\int \sin x \cdot dx = -\cos x$$

$$\int \frac{1}{a^2 + x^2} \cdot dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$$

$$\int \cos x \cdot dx = \sin x$$

$$\int \frac{1}{a^2 - x^2} \cdot dx = \frac{1}{2a} \log \left(\frac{a+x}{a-x} \right)$$

$$\int \sec^2 x \cdot dx = \tan x$$

$$\int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a} \log \left(\frac{x-a}{x+a} \right)$$

$$\int \sec x \cdot \tan x \cdot dx = \sec x.$$

$$\int \frac{f'(x)}{f(x)} \cdot dx = \log f(x).$$

$$\int \tan x \cdot dx = \log (\sec x)$$

$$\int \frac{f'(x)}{\sqrt{f(x)}} \cdot dx = 2 \sqrt{f(x)}$$

$$\int \cot x \cdot dx = \log (\sin x)$$

$$\int \frac{f'(x)}{1 + (f(x))^2} \cdot dx = \tan^{-1} (f(x))$$

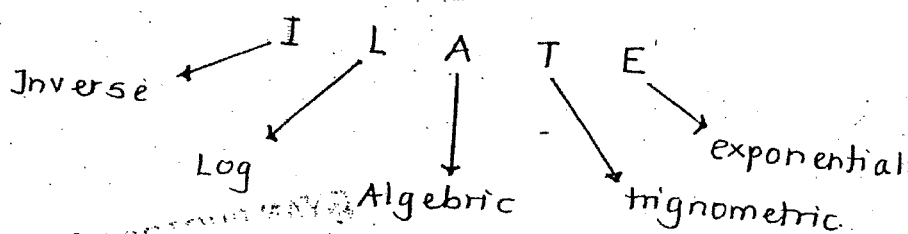
$$\int f(x)^n \cdot f'(x) \cdot dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$\int e^{f(x)} \cdot f'(x) \cdot dx = e^{f(x)}$$

$$\int e^x [f(x) + f'(x)] = e^x \cdot f(x)$$

$$\int f(x) \cdot g(x) \cdot dx = f(x) \int g(x) \cdot dx - \int [f'(x) \cdot \int g(x) \cdot dx] \cdot dx$$

To choose $f(x)$, remember the world.



$$\int e^{ax} \cdot \sin bx \cdot dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$$

$$\int e^{ax} \cdot \cos bx \cdot dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$$

Examples:

shortcut 1:

$$\int \underline{x} \cdot e^x dx = e^x (\underline{x} - 1)$$

$$\int \underline{x^2} \cdot e^x dx = e^x (\underline{x^2} - 2\underline{x} + 2) \frac{d}{dx}$$

$$\int \underline{x^3} \cdot e^x dx = e^x (\underline{x^3} - 3\underline{x^2} + 6\underline{x} - 6) \frac{d}{dx}$$

$$\int (x^2 + x) \cdot e^x dx = e^x [(x^2 + x) - (2x + 1) + 2]$$

shortcut 2:

$$\int x \cdot e^{ax} \cdot dx = e^{ax} \left[\frac{x}{a} - \frac{1}{a^2} \right]$$

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$$\int x^2 \cdot e^{ax} \cdot dx = e^{ax} \left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right]$$

shortcut 3:

$$\int \log x \cdot dx = x (\log x - 1)$$

$$\int (\log x)^2 \cdot dx = x \left[(\log x)^2 - 2 (\log x) + 2 \right]$$

$$\int (\log x)^3 \cdot dx = x \left[(\log x)^3 - 3 (\log x)^2 + 6 (\log x) + 6 \right]$$

1. $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} \cdot dx$

$$1 = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} \cdot dx$$

$$= \left[\sin^{-1}(x) \right]_0^{1/2}$$

$$= \sin^{-1}(1/2) - \sin(0)$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

2. $\int_{\log 2}^{\log 3} \frac{e^x}{1+e^x} \cdot dx$

While evaluating integration of rational function always take derivative of denominator.

$$1 = \left[\log(1+e^x) \right]_{\log 2}^{\log 3}$$

$$= \log(1+e^{\log 3}) - \log(1+e^{\log 2})$$

$$= \log 4 - \log 3 = \log\left(\frac{4}{3}\right)$$

$$3. \int_0^1 \frac{1}{e^x + e^{-x}} \cdot dx$$

$$J = \int_0^1 \frac{e^x}{e^{2x} + 1} dx$$

$$= \frac{1}{2} \int_0^1 \frac{2 \cdot e^x}{e^{2x} + 1} dx \quad (\text{WRONG})$$

$$= \int_0^1 \frac{e^x}{1 + (e^x)^2} dx$$

$$= [\tan^{-1}(e^x)]_0^1$$

$$= \tan^{-1}(e^1) - \tan^{-1}(e^0)$$

$$= \tan^{-1}e - \tan^{-1}(1)$$

$$= \tan^{-1}e - \frac{\pi}{4}$$

$$4. \int_0^1 x \cdot e^x \cdot dx$$

$$J = \int_0^1 x e^x dx$$

$$= [e^x(x-1)]_0^1$$

$$= [x \cdot e^x - e^x]_0^1$$

$$= e^1 - e^1 - (0 - e^0)$$

$$= 1$$

$$5. \int_2^7 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{9-x}} \cdot dx$$

$$J = \int_2^7 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{9-x}} dx$$

$$\int_a^b \frac{f(x)}{f(x) + f(a+b-x)} = \frac{b-a}{2}$$

$$J = \frac{7-2}{2}$$

$$= \frac{5}{2}$$

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6. $\int_0^2 |1-x| \cdot dx$

$$J = \int_0^2 |1-x| \cdot dx$$

check the modulus, for given interval i.e. $(0, 2)$.

If the function totally positive or totally negative then don't split, if it is partially positive and partially negative then split the function.

$$J = \int_0^1 |1-x| \cdot dx + \int_1^2 |1-x| \cdot dx$$

$$|x| = x \text{ if } x \text{ is } +ve$$

$$|x| = -x \text{ if } x \text{ is } -ve$$

$$= \int_0^1 (1-x) \cdot dx + \int_1^2 -(1-x) \cdot dx$$

$$= \left[x - \frac{x^2}{2} \right]_0^1 - \left[x - \frac{x^2}{2} \right]_1^2$$

$$= \left[1 - \frac{1}{2} \right] - \left[2 - \frac{4}{2} - 1 + \frac{1}{2} \right]$$

$$= \frac{+1}{2} - \left[\frac{-2}{4} \right]$$

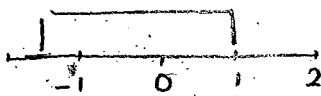
$$= J$$

$$\int_{-1.5}^1 (x+1) \cdot dx$$

step 2 : $[x]$ -

In step function,

If integer lies between given limits, split the function



$$[x+1] =$$

for $-1.5 < x < -1$,

Take $-1.3 \Rightarrow (-1.3+1) = (-0.3)$
(nearest smaller integer is -1)

for $-1 < x < 0$

Take $-0.5 \Rightarrow (-0.5+1) = 0.5$
(nearest smaller integer to 0.5 is 0)

for $0 < x < 1$

Take $0.5 \Rightarrow (0.5+1) = 1.5$
(nearest smaller integer to 1.5 is 1)

$$I = \int_{-1.5}^{-1} -1 \cdot dx + \int_{-1}^0 0 \cdot dx + \int_0^1 1 \cdot dx$$

$$= -[x]_{-1.5}^{-1} + 0 + [x]_0^1$$

$$= -[-1 - (-1.5)] + (1 - 0)$$

$$= 0.5$$

$$2. \int_1^2 x \cdot [x] \cdot dx$$

$$I = \int_1^2 x(1) \cdot dx$$

$$= \left[\frac{x^2}{2} \right]_1^2$$

$$= \frac{3}{2}$$

$$\int_{-1}^1 \frac{1}{1+x^2} \cdot dx$$

(49)

$$I = \int_{-1}^1 \frac{1}{1+x^2} \cdot dx$$

$$\therefore f(x) = \frac{1}{1+x^2} = \frac{1}{1+(-x)^2} = \frac{1}{1+x^2} \text{ i.e. even.}$$

$$I = 2 \int_0^1 \frac{1}{1+x^2} \cdot dx$$

$$= 2 (\tan^{-1} x)_0^1$$

$$= 2 [\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= 2 \left[\frac{\pi}{4} - 0 \right]$$

$$= \frac{\pi}{2}$$

$$\int_{-a}^a \sqrt{\frac{a+x}{a-x}} \cdot dx$$

$$I = \int_{-a}^a \sqrt{\frac{a+x}{a-x}} \cdot dx$$

$$f(x) = \sqrt{\frac{a+x}{a-x}}$$

$$f(-x) = \sqrt{\quad} ?$$

It is difficult to state whether it is even or odd, here.

$$I = \int_{-a}^a \sqrt{\frac{(a+x) \cdot (a+x)}{(a-x) \cdot (a+x)}}$$

$$= \int_{-a}^a \frac{a+x}{\sqrt{a^2-x^2}}$$

$$= \int_{-a}^a \frac{a}{\sqrt{a^2-x^2}} + \int_{-a}^a \frac{x}{\sqrt{a^2-x^2}}$$

Even

Odd.

$$J = 2 \int_0^a \frac{a}{\sqrt{a^2 - x^2}} dx$$

$$= 2a \cdot \left[\sin^{-1} \left(\frac{x}{a} \right) \right]_0^a$$

$$= 2a \left[\sin^{-1} \left(\frac{a}{a} \right) - \sin^{-1} \left(\frac{0}{a} \right) \right]$$

$$= 2a \left[\frac{\pi}{2} - 0 \right]$$

$$= a\pi$$

$$\int_{-a}^a \sqrt{\frac{a+x}{a-x}} = \int_{-a}^a \sqrt{\frac{a-x}{a+x}} = a\pi$$

$$* \int_{-1}^1 \sqrt{\frac{1+x}{1-x}} = \pi$$

TATZOTCHU MAYO

$$\int_{-\pi}^{\pi} x^4 \cdot \sin^5 x \cdot dx$$

$$J = \int_{-\pi}^{\pi} x^4 \cdot \sin^5 x \cdot dx$$

$$f(x) = x^4 \cdot \sin^5 x$$

$$f(-x) = (-x)^4 \cdot \sin^5(-x)$$

$$= -x^4 \sin^5(x)$$

$$J = 0$$

$$\sin(-\theta) = -\sin \theta$$

$$\int_{-\pi}^{\pi} \cos^4 x \cdot \sin^5 x =$$

$$J = \int_{-\pi}^{\pi} \cos^4 x \cdot \sin^5 x = 0$$

$$f(x) = \cos^4 x \cdot \sin^5 x$$

$$f(-x) = \cos^4(-x) \cdot \sin^5(-x)$$

$$= -\cos^4 x \cdot \sin^5 x$$

even
odd

$$\int_0^{\pi/2} \sin^5 x \cdot dx :$$

2, 3, 4, 5

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$$\int_0^{\pi/2} \sin^5 x \cdot dx = \frac{4 \times 2}{5 \times 3 \times 1} \times 1$$

$$= \frac{8}{15}$$

if power is odd

$$\int_0^{\pi/2} \cos^7 x \cdot dx$$

odd power

$$\int_0^{\pi/2} \cos^7 x \cdot dx = \frac{6 \times 4 \times 2}{7 \times 5 \times 3 \times 1} \times 1$$

$$= \frac{16}{35}$$

$$\int_0^{\pi/2} \sin^8 x \cdot dx$$

even

$$\int_0^{\pi/2} \sin^8 x \cdot dx = \frac{7 \times 5 \times 3 \times 1}{8 \times 6 \times 4 \times 2} \times \frac{\pi}{2}$$

$$= \frac{35\pi}{256}$$

1, 2, 3, 4, 5, 6, 7, 8

$$\int_0^1 \frac{x^6}{\sqrt{1-x^2}} dx$$

$$\text{If } (1-x^2) \text{ put } x = \sin \theta$$

$$\text{If } (1+x^2) \text{ put } x = \tan \theta \text{ always.}$$

put

$$x = \sin \theta$$

$$dx = \cos \theta \cdot d\theta$$

$$\text{If } x=0,$$

$$\theta = 0$$

$$x=1$$

$$\theta = \frac{\pi}{2}$$

$$I = \int_0^{\pi/2} \frac{\sin^6 \theta \cdot \cos \theta}{\sqrt{1-\sin^2 \theta}} \cdot d\theta$$

$$= \int_0^{\pi/2} \sin^6 \theta \cdot d\theta$$

$$= \frac{5 \times 3 \times 1}{6 \times 4 \times 2} \times \frac{\pi}{2}$$

$$= \frac{15 \pi}{96}$$

$$= \frac{15}{32} \pi$$

$$\int_0^{\pi/2} \sin^4 x \cdot \cos^5 x \cdot dx$$

$$\int_0^{\pi/2} \sin^4 x \cdot \cos^5 x \cdot dx = \frac{(3 \times 1) \times (4 \times 2 \times 1)}{(4+5) \times 7 \times 5 \times 3 \times 1} \times 1$$

for 4 (even) for 5 (odd)

one power is odd

$$= \frac{24}{315}$$

$$\int_0^{\pi/2} \sin^6 x \cdot \cos^4 x \cdot dx$$

$$\int_0^{\pi/2} \sin^4 x \cdot \cos^4 x \cdot dx = \frac{(5 \times 3 \times 1) \times (3 \times 1)}{10 \times 8 \times 6 \times 4 \times 2} \times \frac{\pi}{2}$$

$$= \frac{3}{512} \frac{\pi}{2}$$

$$\int_0^{\pi/2} \sin^3 x \cdot \cos^3 x \cdot dx$$

$$\int_0^{\pi/2} \sin^3 x \cdot \cos^5 x = \frac{(2 \times 1) \times (4 \times 2)}{8 \times 6 \times 4 \times 2}$$

$$= \frac{1}{24}$$

$$\int_0^{\pi} \sin^3 x \cdot dx$$

$$\int_0^{2a} f(x) \cdot dx = 2 \int_0^a f(x) \cdot dx, \quad f(2a-x) = f(x)$$

$$\begin{aligned}\therefore \int_0^{\pi} \sin^3 x \, dx &= 2 \int_0^{\pi/2} \sin^3 x \, dx \\ &= 2 \times \left(\frac{2}{3 \times 1} \right) \\ &= \frac{4}{3}\end{aligned}$$

Improper integrals:

An integral $\int_a^b f(x) \, dx$ is said to be an improper integral if

- i) $f(x)$ becomes infinite in interval of integration (a, b)
- ii) one or both of limits are infinite ($a = \infty$ or $b = \infty$)

convergent - finite value of integral

divergent - infinite value of integral.

e.g.

$$\begin{aligned}1. \quad \int_0^1 \frac{1}{x} \, dx &= [\log x]_0^1 \\ &= \log 1 - \log 0 \quad \log 0 - \text{undefined} \\ &= \log \left(\frac{1}{0} \right) \\ &= \infty \quad - \text{divergent}\end{aligned}$$

$$\begin{aligned}2. \quad \int_0^1 \sqrt{\frac{(1+x)}{(1-x)}} \, dx &= \int_0^1 \sqrt{\frac{(1+x)(1+x)}{(1-x)(1+x)}} \, dx \\ &= \int_0^1 \frac{1+x}{\sqrt{1-x^2}} \, dx \\ &= \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx + \int_0^1 \frac{x}{\sqrt{1-x^2}} \, dx\end{aligned}$$

$$= \left[\sin^{-1} x \right]_0^1 + \left[\frac{2x\sqrt{1-x^2}}{2} \right]_0^1$$

$$\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)}$$

$$= \left[\sin^{-1} 1 - \sin^{-1} 0 \right] - \left[\sin^{-1} 0 - \sqrt{1-0} \right]$$

$$= \frac{\pi}{2} - 0 - 0 + 1$$

$$= \frac{\pi}{2} + 1$$

$$\int_1^{\infty} \frac{1}{x^3} dx$$

$$I = \int_1^{\infty} x^{-3} dx$$

$$= \left[\frac{x^{-3+1}}{-3+1} \right]_1^{\infty}$$

$$= \left[\frac{x^{-2}}{-2} \right]_1^{\infty}$$

$$= \left[\frac{-1}{2x^2} \right]_1^{\infty}$$

$$= \frac{-1}{2} \left[\frac{1}{\infty} - \frac{1}{1} \right]$$

$$= \frac{-1}{2} [0 - 1]$$

$$= \frac{1}{2}$$

$$\int_{-\infty}^0 \sinh x dx$$

$$I = \left[\cosh x \right]_{-\infty}^0$$

$$= \left[\frac{e^x + e^{-x}}{2} \right]_{-\infty}^0$$

$$= \left[\frac{e^0 + e^{-0}}{2} \right] - \left[\frac{e^{-\infty} + e^{\infty}}{2} \right]$$

$$= 1 - \infty$$

$$= -\infty$$

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$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

$$f(x) = \frac{1}{1+x^2} \quad \text{--- even}$$

$$1 = 2 \int_0^{\infty} \frac{1}{1+x^2} dx$$

$$= 2 \left[\tan^{-1}(x) \right]_0^{\infty}$$

$$= 2 \left[\tan^{-1}(\infty) - \tan^{-1}(0) \right]$$

$$= 2 \left[\frac{\pi}{2} - 0 \right]$$

$$= \pi$$

$$\int_0^{\infty} x \cdot e^{-x^2} dx$$

$$f(x) = -x^2$$

$$f'(x) = -2x$$

$$1 = \int_0^{\infty} x \cdot e^{-x^2} dx$$

$$= \frac{-1}{2} \int_0^{\infty} x \cdot e^{-x^2} \cdot (-2) \cdot dx$$

$$= -\frac{1}{2} \left(e^{-x^2} \right)_0^{\infty}$$

$$= -\frac{1}{2} \left[e^{-\infty} - e^0 \right]$$

$$= -\frac{1}{2} \left[0 - 1 \right]$$

$$= \frac{1}{2}$$

$$\int e^{f(x)} \cdot f'(x) = e^{f(x)}$$

$$\int_0^{\infty} e^{-x^2} dx$$

put, $t = x^2$

$$dt = 2x \cdot dx$$

$$\Rightarrow x = \sqrt{t}$$

$$dx = \frac{dt}{2x}$$

$$= \frac{dt}{2\sqrt{t}}$$

$$x = \infty$$

$$t = \infty$$

$$x = 0$$

$$t = 0$$

$$1 = \int_0^{\infty} e^{-t} \cdot \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2} \int_0^{\infty} e^{-t} \cdot t^{-1/2} dt$$

$-\frac{1}{2}$ - rational no.

$\therefore$ shortcut not valid

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

- gamma function.

$$\Gamma(n+1) = n \Gamma n \quad \text{if } n > 0$$

$$\Gamma(n+1) = n! \quad \text{if } n \in \mathbb{N}$$

$$\Gamma 1 = 1$$

$$\Gamma_{1/2} = \sqrt{\pi}$$

$$1 = \frac{1}{2} \Gamma_{1/2}$$

$$= \frac{\sqrt{\pi}}{2}$$

$$\therefore 1 = \int_0^{\infty} e^{-t} t^{1/2-1} dt$$

$$\therefore \Gamma_{1/2} = \sqrt{\pi}$$

$$\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy \quad [\text{standard}]$$

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$$J = \int_0^{\infty} \int_0^{\infty} e^{-x^2} \cdot e^{-y^2} dx dy$$

$$= \int_0^{\infty} e^{-x^2} dx \cdot \int_0^{\infty} e^{-y^2} dy$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \frac{\pi}{4}$$

$$\int_0^{\infty} \frac{x}{(x^2+g)^2} dx$$

$$J = \int_0^{\infty} (x^2+g)^{-2} \cdot x \cdot dx$$

$$= \frac{1}{2} \int_0^{\infty} 2x \cdot (x^2+g)^{-2} dx$$

$$= \frac{1}{2} \left[\frac{(x^2+g)^{-2+1}}{-2+1} \right]_0^{\infty}$$

$$\therefore \int [f(x)]^n \cdot f'(x) \cdot dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= \left[\frac{1}{2} \frac{(x^2+g)^{-1}}{-1} \right]_0^{\infty}$$

$$= \left[\frac{-1}{2(x^2+g)} \right]_0^{\infty}$$

$$= \left[\frac{-1}{2(\infty+g)} - \frac{1}{2} \frac{1}{(g)} \right] - \left[\frac{-1}{2(0+g)} - \frac{-1}{2} \right]$$

$$= -\frac{1}{2} \left(0 - \frac{1}{g} \right) = \frac{1}{18}$$

vector calculus :

Gradient of scalar function :

Let $\phi(x, y, z)$ be any scalar function then the gradient of ϕ is denoted by $\text{grad } \phi$ or $\nabla \phi$ and is defined as

$$\text{grad } \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

We know that,

$$\vec{r} = xi + yj + zk$$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$2r \cdot \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

similarly

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\therefore \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\begin{aligned} \text{grad } r &= \nabla r = i \frac{\partial r}{\partial x} + j \frac{\partial r}{\partial y} + k \frac{\partial r}{\partial z} \\ &= i \left(\frac{x}{r} \right) + j \left(\frac{y}{r} \right) + k \left(\frac{z}{r} \right) \\ &= \frac{xi + yj + zk}{r} \\ &= \frac{\vec{r}}{r} \end{aligned}$$

$$\text{grad } r = \nabla r = \frac{\vec{r}}{r}$$

∇ is vector differential operator

e.g.

$$(i) \nabla \log r = \frac{1}{r} \nabla r = \frac{1}{r} \cdot \frac{\vec{r}}{r} = \frac{\vec{r}}{r^2}$$

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$$(ii) \nabla \frac{1}{r} = -\frac{1}{r^2} \nabla r = -\frac{1}{r^2} \cdot \frac{\vec{r}}{r} = -\frac{\vec{r}}{r^3}$$

$$(iii) \nabla r^n = n \cdot r^{n-1} \cdot \nabla r = n \cdot r^{n-1} \cdot \frac{\vec{r}}{r} = n \cdot r^{n-2} \cdot \vec{r}$$

Tangent vector to a curve:

Let $\vec{r}(t)$ be the given vector curve then $\frac{d\vec{r}}{dt}$ is called the tangent vector to the given curve.

$$\vec{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k}$$

Normal to a surface:

Let $\phi(x, y, z) = c$ be any surface, the $\nabla(\phi)$ is called normal to the surface ϕ and $\frac{\nabla \phi}{|\nabla \phi|}$ is unit normal vector to the surface ϕ and is denoted by N

$$N = \frac{\nabla \phi}{|\nabla \phi|}$$

Directional derivative:

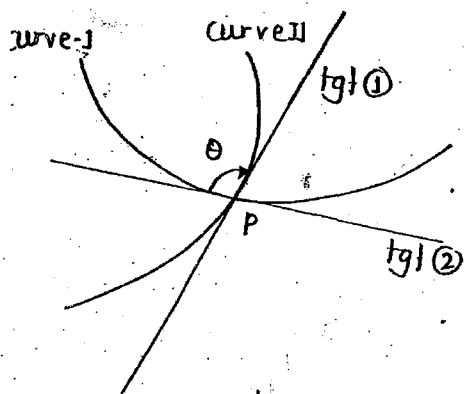
Let $\phi(x, y, z) = c$ be any surface, then directional derivative of ϕ is defined by $\nabla \phi \cdot e$ where e is the unit vector in the direction of given vector.

If given vector is a

$$e = \frac{\vec{a}}{|\vec{a}|}$$

$$\text{directional derivative} = \nabla \phi \cdot e$$

The max. or greatest value of directional derivative or magnitude of directional derivative of a scalar function ϕ is defined by $|\nabla \phi|$.



For tangent ①

$$a = a_1 i + a_2 j + a_3 k$$

For tangent ②

$$b = b_1 i + b_2 j + b_3 k$$

$$\therefore \cos \theta = \frac{a \cdot b}{|a| |b|}$$

where,

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Note:

The two curves cut each other orthogonally ($\theta = 90^\circ$)
 $a \cdot b = 0$

Angle between two surfaces: -

The angle between the two surfaces is the angle between the normals drawn at their point of intersection.

If f and g are any two surfaces and θ is angle between them,

$$\cos \theta = \frac{\nabla f \cdot \nabla g}{|\nabla f| \cdot |\nabla g|}$$

Note:

If the two surfaces cut each other orthogonally, then

$$\nabla f \cdot \nabla g = 0$$

Q. Find the unit normal vector to the surface

$$x^3 + y^3 + 3xyz = 3 \quad \text{at } (1, 2, -1)$$

(50)

$$\nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

$$= i(3x^2 + 3yz) + j(3y^2 + 3xz) + k(3xy)$$

$$(\nabla \phi)_{(1,2,-1)} = -3i + 9j + 6k$$

$$N = \frac{\nabla \phi}{|\nabla \phi|}$$

$$= \frac{3(-i + 3j + 2k)}{3\sqrt{1+9+4}}$$

$$= \frac{-i + 3j + 2k}{\sqrt{14}}$$

Q. Find unit normal vector for the surfaces.

$$x^2y + 2xz = 4 \quad \text{at } (2, -2, 3)$$

$$\nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

$$= i(2xy + 2z) + j(2x) + k(2x)$$

$$= i(-4 + 6) + j(4) + k(4)$$

$$= +2i + 4j + 4k$$

$$N = \frac{-2(i + 2j + 2k)}{\sqrt{4+16+16}}$$

$$= \frac{-i + 2j + 2k}{\sqrt{9}}$$

$$= \frac{-i + 2j + 2k}{3}$$

$$x^2 + y^2 + z^2 = 1 \quad \text{at} \quad \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

a) $\frac{i+j}{2}$

b) $\frac{j+k}{\sqrt{2}}$

c) $\frac{i+k}{\sqrt{2}}$

d) $\frac{i+j+k}{\sqrt{3}}$

Ans: (c) $\frac{i+k}{\sqrt{2}}$

Find the max. value of directional derivative to the surface
 $\phi = x^2 y \cdot z^3$ at $(2, 1, -1)$

$$\nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

$$= i(2xy \cdot z^3) + j(x^2 z^3) + k(3x^2 y \cdot z^2)$$

$$= i(-4) + j(-4) + k(12)$$

$$= -4(i + j - 3k)$$

$$|\nabla \phi| = \sqrt{16 + 16 + 144}$$

$$= \sqrt{176}$$

Find the greatest value of directional derivative to surface
 $\phi = x^2 y z$ at $(1, 4, 1)$

$$\nabla \phi = i(2xyz) + j(x^2 z) + k(x^2 y)$$

$$= 8i + 4j + 4k$$

$$|\nabla \phi| = \sqrt{64 + 16 + 16}$$

$$= \sqrt{96}$$

$$= 9.8$$

Find magnitude of gradient $u = \frac{x^2}{2} + \frac{y^2}{3}$ at $(1, 3)$ Q6

$$\nabla u = \frac{2x}{2} \cdot i + \frac{2y}{3} \cdot j + 0k$$

$$= 1i + \frac{2}{3} \times 3j$$

$$= i + 2j$$

$$|\nabla u| = \sqrt{1+4}$$

$$= \sqrt{5}$$

Find directional derivative of $f = xy + yz + zx$ at $(1, 2, 0)$ in the direction of $i + 2j + 2k$.

$$\nabla f = i(y+z) + j(x+z) + k(x+y)$$

$$= i(2) + j(2) + k(3)$$

$$= 2j + j2 + k \times 3$$

$$e = \frac{i + 2j + 2k}{\sqrt{1+4+4}}$$

$$= \frac{i + 2j + 2k}{3}$$

$$\text{Directional derivative} = \nabla f \cdot e$$

$$= 2 \times \frac{1}{3} + 1 \times \frac{2}{3} + 3 \times \frac{2}{3}$$

$$= \frac{10}{3}$$

Find directional derivative of $f = 2xy + z^2$ at $(1, -1, 3)$ in direction of $i + 2j + 3k$.

$$\nabla f = i(2y) + j(2x) + k(2z)$$

$$= -2j + 2j + 6k$$

$$e = \frac{i + 2j + 3k}{\sqrt{1+4+9}} = \frac{i + 2j + 3k}{\sqrt{14}}$$

$$\Delta f \cdot e = -2 \times \frac{1}{\sqrt{14}} + 2 \times \frac{2}{\sqrt{14}} + 6 \times \frac{3}{\sqrt{14}} = \frac{20}{\sqrt{14}}$$

Find directional derivative of $f(x^2 - y^2 + 2z^2)$ at $P(1, 2, 3)$ in the direction of $\overline{PQ}$ where $Q(5, 0, 4)$

$$a = \overline{PQ} = Q - P$$

$$= 4i - 2j + k$$

$$e = \frac{4i - 2j + k}{\sqrt{16 + 4 + 1}}$$

$$= \frac{4i - 2j + k}{\sqrt{21}}$$

$$\nabla f = i(2x) + j(-2y) + k(4z)$$

$$= 2i - 4j + 12k$$

$$\nabla f \cdot e = \left(2 \times \frac{4}{\sqrt{21}}\right) + \left(-4 \times \frac{-2}{\sqrt{21}}\right) + \left(12 \times \frac{1}{\sqrt{21}}\right)$$

$$= \frac{28}{\sqrt{21}}$$

Find directional derivative of $f = xy + yz + zx$ at $P(1, 2, -1)$ in the direction of $\overline{PQ}$ where $Q(1, 2, 3)$

$$a = \overline{PQ} = Q - P$$

$$= 0i + 0j + 4k$$

$$e = \frac{4k}{4}$$

$$= k$$

$$\nabla f = i(y+z) + j(x+z) + k(x+y)$$

$$= i(1) + j(0) + k(3)$$

$$= i + 3k$$

$$\nabla f \cdot e = 3 \times 1$$

$$= 3$$

Find the directional derivative of $f = xy^2 + yz^2 + zx^2$ at $P(1, 1, 1)$ along the tangent to curve $x=t, y=t^2, z=t^3$ (5)

$$a = \frac{d\vec{r}}{dt} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k}$$

$$= \mathbf{i} + 2t \mathbf{j} + 3t^2 \mathbf{k}$$

$$e = \frac{a}{|a|}$$

$$= \frac{\mathbf{i} + 2t \mathbf{j} + 3t^2 \mathbf{k}}{\sqrt{1 + 4t^2 + 9t^4}}$$

Take $x=t$ (or $y=t^2$ or $z=t^3$)

$$t=1$$

$$= \frac{\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}}{\sqrt{1+4+9}}$$

$$= \frac{\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}}{\sqrt{14}}$$

$$\nabla f = (y^2 + 2xz) \mathbf{i} + (2xy + 2z^2) \mathbf{j} + k(2yz + x^2)$$

$$\nabla f_{(1,1,1)} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

$$\text{directional derivative} = 3 \left(\frac{1}{\sqrt{14}} \right) + 3 \left(\frac{2}{\sqrt{14}} \right) + 3 \left(\frac{3}{\sqrt{14}} \right)$$

$$= \frac{18}{\sqrt{14}}$$

Find the directional derivative of $f = xyz^2 + xz$ at $(1, 1, 1)$ in the direction of normal of $g = 3xy^2 + y - z$ at $(0, 1, 1)$

$$a = \nabla g$$

$$= \mathbf{i}(3y^2) + \mathbf{j}(6xy + 1) + \mathbf{k}(-1)$$

$$= 3\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$e = \frac{3\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{11}}$$

$$\nabla f = i(yz^2 + z) + j(xz^2) + k(2xyz + x)$$

$$= 2i + j + 3k$$

Directional derivative

$$= \frac{3}{\sqrt{11}} \times 2 + \frac{1}{\sqrt{11}} \times 1 - \frac{1}{\sqrt{11}} \times 3$$

$$= \frac{4}{\sqrt{11}}$$

Thursday
26th September 2013

2. Find the angle between $x^2 + y^2 + z^2 = 9$, $x^2 + y^2 - z = 3$ at $(2, -1, 2)$

$$\Delta f = 2xi + 2yj + 2zk$$

$$= 4i - 2j + 4k$$

$$\Delta g = 2xi + 2yj \pm k$$

$$= 4i - 2j - k$$

$$\cos \theta = \frac{\nabla f \cdot \nabla g}{|\nabla f| |\nabla g|}$$

$$= \frac{16 + 4 - 4}{\sqrt{16 + 4 + 16} \sqrt{16 + 4 + 1}}$$

$$\theta = \cos^{-1} \left(\frac{8}{3\sqrt{21}} \right)$$

Q. Find acute angle between $xy^2z = 3x + z^2$ & $3x^2 - y^2 + 2z = 1$ at $(1, -2, 1)$

$$\nabla f = (y^2z - 3)i + j(2xyz) + k(xy^2 - 2z)$$

$$= i - 4j + 2k$$

$$\nabla g = (6x)i + j(-2y) + zk$$

$$= 6xi + 4j + 2k$$

$$\cos \theta = \frac{|\nabla f \cdot \nabla g|}{|\nabla f| \cdot |\nabla g|}$$

$$= \frac{|16 - 16 + 4|}{\sqrt{1+16+4} \sqrt{36+16+4}}$$

$$\theta = \cos^{-1}\left(\frac{3}{7\sqrt{6}}\right)$$

Q. Find the constants so that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at $(1, -1, 2)$

$$f = ax^2 - byz - (a+2)x$$

$$\nabla f = (2ax)i - (a+2)k$$

$$= (2ax - a - 2)i - (by)j - (bz)k$$

$$\nabla f_{(1,-1,2)} = (a-2)i - 2bj + bk$$

$$\nabla g = (8xy)i + (4x^2)j + (3z^2)k$$

$$= -8i + 4j + 12k$$

$$\nabla f \cdot \nabla g = (a-2) \times (-8) - 2b \times 4 + 12 \times b$$

$$0 = -8a + 16 - 8b + 12b$$

$$= 16 + 4b - 8a$$

$$b = 2a - 4 \quad \text{--- (i)}$$

put in 2nd surface

$$a(1)^2 - b(-1)(2) = (a+2) \cdot 1$$

$$a + 2b = (a+2)$$

$$b = 1$$

$$2a = b + 4$$

$$a = \frac{5}{2}$$

Divergence of a vector function:

Let $F = F_1 i + F_2 j + F_3 k$ be any differentiable vector function then divergence of F is denoted by $\text{div} \cdot F$ or $\nabla \cdot F$ and is defined as

$$\text{div} \cdot F \text{ or } \nabla F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$F = xi + yj + zk$$

$$\text{div} \cdot F = 1 + 1 + 1$$

$$\therefore * \text{div} F = \nabla \cdot F = 3$$

Solenoidal vector:

The necessary and sufficient condition for a vector function F to be solenoid is $\text{div} \cdot F = 0$
i.e.

$$\nabla \cdot F = 0$$

Note:

Let ϕ is scalar and A is vector then:

$$\text{div} \cdot (\phi \cdot A) = (\text{grad } \phi) \cdot A + \phi \cdot \text{div} \cdot (A)$$

$$\nabla \cdot (\phi A) = (\nabla \phi) \cdot A + \phi \cdot (\nabla \cdot A)$$

(product of scalar and vector w.r.t divergence)

e.g.

$$\nabla^2 \frac{1}{r}$$

$$= \nabla \cdot \nabla \cdot \frac{1}{r}$$

$$= \nabla \cdot \left(\frac{-1}{r^2} \cdot \nabla r \right)$$

$$= \nabla \cdot \left(\frac{-1}{r^2} \cdot \frac{\vec{r}}{r} \right)$$

$$= \nabla \cdot \left(\frac{-1}{r^3} \cdot \vec{r} \right)$$

scalar

vector

$$\therefore \nabla r = \frac{\vec{r}}{r}$$

$$\begin{aligned}
 \therefore \nabla^2 \frac{1}{r} &= \nabla \cdot \left(\frac{-1}{r^3} \right) \vec{r} + \left(\frac{-1}{r^3} \right) \nabla \cdot \vec{r} \\
 &= \frac{3}{r^4} \cdot \nabla \vec{r} \cdot \vec{r} - \frac{1}{r^3} \times 3 \\
 &= \frac{3}{r^4} \cdot \frac{\vec{r}}{r} \cdot \vec{r} - \frac{3}{r^3} \\
 &= \frac{3}{r^4} \cdot \frac{r^2}{r} - \frac{3}{r^3} \\
 &= \frac{3}{r^3} - \frac{3}{r^3} \\
 &= 0
 \end{aligned}$$

$\therefore \nabla \cdot \vec{r} = 3$
 $\therefore \vec{r} \cdot \vec{r} = r^2$
 $\therefore \nabla r = \frac{\vec{r}}{r}$

Shortcut 1:

$$\nabla^2 [f(r)] = f''(r) + \frac{2}{r} f'(r)$$

In above problem,

$$f(r) = \frac{1}{r}, \quad f'(r) = \frac{-1}{r^2}, \quad f''(r) = \frac{2}{r^3}$$

$$\begin{aligned}
 \therefore \nabla^2 \left(\frac{1}{r} \right) &= \frac{2}{r^3} + \frac{2}{r} \left(\frac{-1}{r^2} \right) \\
 &= \frac{2}{r^3} - \frac{2}{r^3}
 \end{aligned}$$

e.g. 1. $\nabla^2 (\log r)$

$$\begin{aligned}
 &= \frac{-1}{r^2} + \frac{2}{r} \left(\frac{1}{r} \right) \\
 &= \frac{-1}{r^2} + \frac{2}{r^2} \\
 &= \frac{1}{r^2}
 \end{aligned}$$

$$\nabla (r^n)$$

$$\begin{aligned}
 &= n(n-1)r^{n-2} + \frac{2}{r} (n \cdot r^{n-1}) \\
 &= (n^2 - n) \cdot r^{n-2} + \frac{2}{r} (n \cdot r^{n-1}) \\
 &= n(n-1)r^{n-2} + 2 \cdot n \cdot r^{n-2} \\
 &= n(n-1+2)r^{n-2} \\
 &= n(n+1) \cdot r^{n-2}
 \end{aligned}$$

Q. If $\nabla \cdot (r^n \cdot \vec{r}) = 0$ then $n = ?$

$$\nabla \cdot (\phi \cdot A) = (\nabla \phi) \cdot A + \phi \cdot (\nabla \cdot A)$$

$$0 = (\nabla \cdot r^n) \cdot \vec{r} + r^n (\nabla \cdot \vec{r})$$

$$= n \cdot r^{n-1} \cdot \nabla r \cdot \vec{r} + r^n (3)$$

$$= n \cdot r^{n-1} \cdot \frac{\vec{r}}{r} \cdot \vec{r} + 3r^n$$

$$= n \cdot \frac{r^{n-1} \cdot r^2}{r} + 3r^n$$

$$= n \cdot r^2 + 3r^n$$

$$(n+3) = 0$$

$$n = -3$$

Curl of a vector function:

Let $F = F_1 i + F_2 j + F_3 k$ be any differentiable vector function then the curl of F is denoted by $\text{curl } F$ or $\nabla \times F$ and is defined as

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

If

$$\vec{r} = x i + y j + z k$$

$$\nabla \cdot \vec{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= i \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) - j \left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z} \right) + k \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right)$$

$$\nabla \times \vec{r} = 0$$

Note:

— let F be any non-zero vector function, then

$$\text{div} \cdot (\text{curl } F) = 0$$

or

$$\nabla \cdot (\nabla \times F) = 0$$

Q. If $f = x^3 + y^3 + z^3 - 3xyz$ then $\text{div} \cdot (\text{grad } f) = ?$

$$\nabla \cdot (\nabla f) = \nabla^2 f$$

shortcut 2:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 6x + 6y + 6z$$

$$= 6(x+y+z)$$

Q. If $f = (x+2y)i + (y+2z)j + (x+\lambda z)k$ if this function is solenoidal then find λ .

$$\nabla \cdot F = 0$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$

$$1 + 1 + \lambda = 0$$

$$\lambda = -2$$

Q. Find the value of λ so that vector function given is zero divergencer.

$$F = (2\lambda x^2 y + yz)i + (xy^2 - xz^2)j + (2xyz - 2x^2 y^2)k$$

$$\nabla \cdot F = 0$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$

$$4\lambda x y + 2xy + 2xy = 0$$

$$2xy(2\lambda + 2) = 0$$

$$\lambda = -1$$

Q. If $F = 3x^2i + 5xy^2j + xyz^3k$ then $(\text{div } F)_{(2,3)} = ?$

$$\text{div } F = 6x + 10xy + 3xyz^2$$

$$= 6(1) + 10(2) + 3(18)$$

$$= 86$$

Irrotational vector:

The necessary and sufficient condition for a vector function F to be irrotational is $\text{curl } F = \text{zero}$.

$$\text{curl } F = 0$$

$$\nabla \times F = 0$$

- * (i) curl represents angular velocity of the vector field
- (ii) divergence represents normal flux of the vector field.

Q. If $F = xy^2i + 2x^2yzj + 3yz^2k$ then $\text{curl } (F)$ at $(1, -1, 1)$ is

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= i \left(\frac{\partial}{\partial y} (-3yz^2) - \frac{\partial}{\partial z} (2x^2yz) \right)$$

$$- j \left(\frac{\partial}{\partial x} (-3yz^2) - \frac{\partial}{\partial z} (xy^2) \right)$$

$$+ k \left(\frac{\partial}{\partial x} (-3yz^2) - \frac{\partial}{\partial y} (x^2y) \right)$$

$$= i(-2x^2y) - j(0) + k(0 - 2xy)$$

$$= -2x^2y i - 2xy k$$

$$= 2i - 2k$$

If $F = 2xy\mathbf{i} - x^2z\mathbf{j}$ then $\text{curl}(F)$ at $(1, 1, 1)$ is

(6)

$$\nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & -x^2z & 0 \end{vmatrix}$$

$$= \mathbf{i}(0 + x^2) - \mathbf{j}(0) + \mathbf{k}(-2xz - 2x)$$

$$\nabla \times F_{(1,1,1)} = \mathbf{i} - 4\mathbf{k}$$

Q Find the constants a, b, c so that vector function given is irrotational.

$$F = (x + 2y + az)\mathbf{i} + (bx - 3y - z)\mathbf{j} + (4x + cy + 2z)\mathbf{k}$$

$$\nabla \times F = 0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

$$\mathbf{i}[c + 1] - \mathbf{j}[4 - a] + \mathbf{k}[b - 2] = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

$$c + 1 = 0$$

$$c = -1$$

$$4 - a = 0$$

$$a = 4$$

$$b - 2 = 0$$

$$b = 2$$

Q If $F = x^2yz\mathbf{i} + xyz^3\mathbf{j} + 3x^2y^2\mathbf{k}$ then $[\nabla \cdot (\nabla \times F)] =$ at $(1, -2, 1)$

$$\nabla \cdot (\nabla \times F) = 0$$

for any non-zero vector function.

Vector Integration :

- (i) Line integral $(\int)$
- (ii) Surface integral $(\iint)$
- (iii) Volume integral $(\iiint)$

(i) Line integral :

Any integral which is evaluated along the curve is called Line integral.

Let F be any differentiable vector function defined along the curve C then, line integral of F is defined as

$$\int_C F \cdot d\vec{r}$$

If C is a closed curve, it is denoted by $\oint_C F \cdot d\vec{r}$

Cartesian form :

$$\text{Let } F = F_1 i + F_2 j + F_3 k$$

$$\vec{r} = x i + y j + z k$$

$$d\vec{r} = dx \cdot i + dy \cdot j + dz \cdot k$$

$$\therefore \int_C F \cdot d\vec{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

Note :

If x, y, z are the functions in t then,

$$\int_C F \cdot d\vec{r} = \int \left[F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right] dt$$

Q. The value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (2y+3)\mathbf{i} + xz\mathbf{j} + (yz-x)\mathbf{k}$ (6)

where C is

(i) The curve $x=2t^2$, $y=t$, $z=t^3$ joining $(0,0,0)$, $(2,1)$

(ii) The st. line joining $(0,0,0)$ to $(0,0,-1)$ to $(0,1,1)$ to $(2,1,1)$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left[F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right] dt$$

$$x = 2t^2$$

$$y = t$$

$$z = t^3$$

$$\frac{dx}{dt} = 4t$$

$$\frac{dy}{dt} = 1$$

$$\frac{dz}{dt} = 3t^2$$

portion of
curve bound
by points

$(0,0,0)$

$$\text{If } y=0, \quad t=0$$

$$y=1, \quad t=1$$

$(2,1,1)$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left[(2t+3) \cdot 4t + (2t^5) \cdot 1 + (t^4 - 2t^2) \cdot 3t^2 \right] dt$$

$$= \int_0^1 \left[8t^3 + 12t + 2t^5 + 3t^6 - 6t^4 \right] dt$$

$$= \left[8 \cdot \frac{t^4}{4} + 12 \cdot \frac{t^2}{2} + 2 \cdot \frac{t^6}{6} + 3 \cdot \frac{t^7}{7} - 6 \cdot \frac{t^5}{5} \right]_0^1$$

$$= \left[2 \times 1 + 6 \times 1 + \frac{1}{3} + \frac{3}{7} - 6 \times \frac{1}{5} \right]$$

$$= 8 + \frac{10}{21} - \frac{6}{5}$$

$$= \frac{288}{35}$$

$$O \equiv (0, 0, 0)$$

$$A \equiv (0, 0, 1)$$

$$B \equiv (0, 1, 1)$$

$$C \equiv (2, 1, 1)$$

① Along OA

$$x=0$$

$$y=0$$

$$dx=0$$

$$dy=0$$

$$z = 0 \text{ to } 1$$

$$\int_{OA} (yz-x) \cdot dz = 0$$

$$\therefore y=0 \text{ \& } x=0$$

② Along AB

$$x=0,$$

$$z=1$$

$$dx=0$$

$$dz=0$$

$$y = 0 \text{ to } 1$$

$$\int_{AB} x \cdot z \cdot dy = 0$$

$$= 0$$

③ Along BC

$$y=1$$

$$z=1$$

$$dy=0$$

$$dz=0$$

$$x = 0 \text{ to } 2$$

$$\int_{BC} (2y+3) \cdot dx = \int_0^2 (2(1)+3) \cdot dx$$

$$= 5(2-0)$$

$$= 10$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 10$$

$$= 10$$

Q. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 3x^2\mathbf{i} + (2xz - y)\mathbf{j} + z\mathbf{k}$ where C is the straight line joining $(0, 0, 0)$ to $(2, 1, 3)$

① For OA

$$O \equiv (0, 0, 0)$$

$$A \equiv (2, 1, 3)$$

Equⁿ of st. line

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$\therefore$ Equⁿ of st. line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) in the symmetric form is.

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

where t is some scalar quantity.

Equⁿ of OA

$$\frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0} = t$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$$x = 2t$$

$$y = t$$

$$z = 3t$$

$$\frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = 1$$

$$\frac{dz}{dt} = 3$$

$$\text{If } x = 0, \quad t = 0$$

$$x = 2, \quad t = 1$$

$$\int_{t=0}^1 \left[3x^2 \cdot \frac{dx}{dt} + (2xz - y) \cdot \frac{dy}{dt} + z \cdot \frac{dz}{dt} \right] dt$$

$$= \int_0^1 \left[3(2t)^2 \cdot 2 + (2 \times 2t \times 3t - t) \cdot 1 + 3t \times 3 \right] dt$$

$$= \int_0^1 \left[24t^2 + (12t^2 - t) + 9t \right] dt$$

$$= \int_0^1 (36t^2 + 8t) dt$$

$$= \left[36 \left(\frac{t^3}{3} \right) + 8 \left(\frac{t^2}{2} \right) \right]_0^1$$

$$= 36 \times \frac{1}{3} + 8 \times \frac{1}{2}$$

$$= 16$$

Q. Find $\int_c \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 2xyz \mathbf{i} + x^2y \mathbf{j} + x^2z \mathbf{k}$ where c is straight line joining $(0,0,0)$ to $(1,1,1)$

Eqn of straight line -

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 1$$

$$\therefore \int_c \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t^3 + t^3 + t^3) dt$$

$$= \int_0^1 4t^3 dt$$

$$= \frac{4}{4} [1]$$

$$= 1$$

Q. Find the value of $\int_c \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 3xy \mathbf{j} - y^2 \mathbf{j}$ where c is the curve $y = 2x^2$ in the xy -plane joining $(0,0)$ to $(1,2)$

$$\int_C F \cdot d\vec{r} = \int_C (3xy) \cdot dx - y^2 \cdot dy \quad (64)$$

$$y = 2x^2$$

$$dy = 4x \cdot dx$$

put the limits of x i.e. 0 to 1

$$\begin{aligned} \int_C F \cdot d\vec{r} &= \int_0^1 (3x \times 2x^2) \cdot dx - 4x^4 \cdot 4x \cdot dx \\ &= \int_0^1 6x^3 \cdot dx - \int_0^1 16x^5 \cdot dx \end{aligned}$$

$$= \frac{6}{4} [1] - \frac{16}{6} [1]$$

$$= \frac{-7}{6}$$

Q. Find $\int_C F \cdot d\vec{r}$ where $F = (5xy - 6x^2)\mathbf{i} + (2y - 4x)\mathbf{j}$ where C is curve $y = x^3$ in X - Y plane joining $(1,1)$ to $(2,8)$

$$\begin{aligned} \int_C F \cdot d\vec{r} &= \int_C (5xy - 6x^2) \cdot dx + (2y - 4x) \cdot dy \\ y &= x^3 \\ dy &= 3x^2 \cdot dx \end{aligned}$$

$$\begin{aligned} \int_C F \cdot d\vec{r} &= \int_1^2 (5x \cdot x^3 - 6x^2) \cdot dx + (2x^3 - 4x) \cdot 3x^2 \cdot dx \\ &= \int_1^2 (5x^4 - 6x^2) \cdot dx + \int_1^2 (6x^5 - 12x^3) \cdot dx \\ &= \frac{5}{5} [2^5 - 1^5] - \frac{6}{3} [2^3 - 1^3] + \frac{6}{6} [2^6 - 1] - \frac{12}{4} [2^4] \\ &= [32 - 1] - 2[8 - 1] + 1[64 - 1] - 3[16] \end{aligned}$$

2. Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (2x - y + z)\mathbf{i} + (x + y - z^2)\mathbf{j} + (3x - 4y + 2z)\mathbf{k}$ where C is the circle in the xy -plane having centre at origin and of radius 3 units.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (2x - y) dx + (x + y) dy$$

$\therefore xy$ -plane, $z = 0$
 $dz = 0$

Equation of circle,

$$x^2 + y^2 = r^2$$

$$x^2 + y^2 = 9$$

Parametric form of circle,

$$x = 3 \cos \theta, \quad y = 3 \sin \theta$$

$$dx = -3 \sin \theta \cdot d\theta \quad dy = 3 \cos \theta \cdot d\theta$$

Limits will be

$$\theta = 0 \quad \text{to} \quad \theta = 360^\circ$$

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} [2 \times 3 \cos \theta - 3 \sin \theta] \cdot (-3 \sin \theta \cdot d\theta) \\ &\quad + \int_0^{2\pi} (3 \cos \theta + 3 \sin \theta) \cdot 3 \cos \theta \cdot d\theta \end{aligned}$$

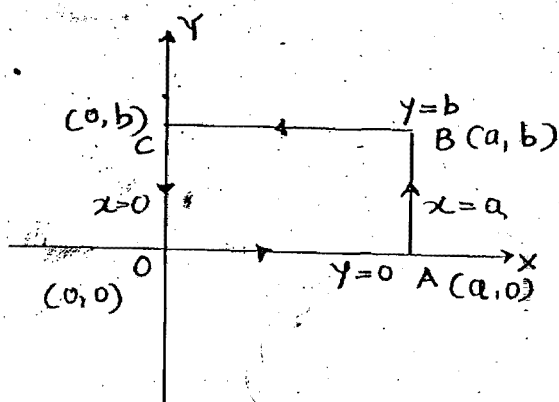
$$= \int_0^{2\pi} (-9 \sin \theta \cdot \cos \theta + 9) \cdot d\theta$$

$$= \int_0^{2\pi} -9 \sin \theta \cdot \cos \theta + 9 \int_0^{2\pi} d\theta$$

$$= +9 \left[\frac{\sin^2 \theta}{2} \right]_0^{2\pi} + \left[9\theta \right]_0^{2\pi}$$

$$= 18\pi$$

Q. Find $\oint_C F \cdot d\vec{r}$ where $F = (x^2 + y^2)\vec{i} + 2xy\vec{j}$ where C is the rectangle bounded by $x=0, x=a$ & $y=0, y=b$. (68)



$$\oint_C F \cdot d\vec{r} = \oint_C (x^2 + y^2) \cdot dx - (2xy) \cdot dy$$

① Along OA

$$y = 0$$

$$dy = 0$$

Limits $x = 0$ to a

$$= \int_0^a (x^2 + 0^2) \cdot dx = \left[\frac{x^3}{3} \right]_0^a$$

$$= \frac{a^3}{3}$$

② Along AB

$$x = a$$

$$dx = 0$$

Limits $y = 0$ to $y = b$

$$= \int_0^b - (2a \cdot y) \cdot dy$$

$$= -2a \left[\frac{y^2}{2} \right] = -a \cdot b^2$$

③ Along BC

$$y = b$$

$$dy = 0$$

Limits $x = 0$ to $x = a$

$$= \int_0^a (x^2 + b^2) \cdot dx$$

$$= \left(\frac{x^3}{3} + b^2 \cdot x \right)_0^a$$

$$= 0 - \left(\frac{a^3}{3} + ab^2 \right)$$

$$= -\frac{a^3}{3} - ab^2$$

④ Along co

$$x=0$$

$$dx=0$$

$$\text{Limits } y=b \text{ to } y=0$$

$$= \int_b^0 - (2 \times 0)$$

$$= 0$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 - 0$$

$$= -2ab^2$$

(2) Surface Integrals:

Any integral which is evaluated over surface is called a surface integral.

Let $\mathbf{F}$ be any differentiable vector function defined over a surface 'S' then the surface integral is defined by $\iint_S \mathbf{F} \cdot \mathbf{N} \cdot d\mathbf{s}$, where $\mathbf{N}$ is the outer unit normal vector to the given surface.

$$\iint_S \mathbf{F} \cdot \mathbf{N} \cdot d\mathbf{s} = \iint_{R_1} \mathbf{F} \cdot \mathbf{N} \cdot \frac{dx \cdot dy}{|\mathbf{N} \cdot \mathbf{k}|}$$

R_1 - projection of S in XY-plane.

$$= \iint_{R_2} \mathbf{F} \cdot \mathbf{N} \cdot \frac{dy \cdot dz}{|\mathbf{N} \cdot \mathbf{j}|}$$

R_2 - projection of S in YZ-plane

$$= \iint_{R_3} \mathbf{F} \cdot \mathbf{N} \cdot \frac{dx \cdot dz}{|\mathbf{N} \cdot \mathbf{i}|}$$

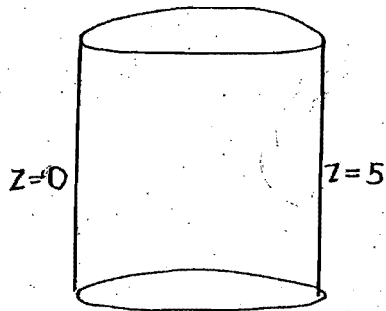
R_3 - projection of S in XZ-plane

$\mathbf{F} \cdot \mathbf{N}$ gives an idea to decide the projection.

Q. Evaluate $\iint_S \mathbf{F} \cdot \mathbf{N} \cdot d\mathbf{s}$ where $\mathbf{F} = z\mathbf{i} + xy\mathbf{j} - 3y^2z\mathbf{k}$ where (66)

S is the surface of the cylinder $x^2 + y^2 = 16$, included in the 1st octant bounded by $z=0$ and $z=5$.

Projection in YZ or XZ plane can be taken.



$$\therefore \phi = x^2 + y^2 - 16$$

$$\nabla \phi = 2x\mathbf{i} + 2y\mathbf{j}$$

$$\begin{aligned} \mathbf{N} &= \frac{\nabla \phi}{|\nabla \phi|} \\ &= \frac{2(x\mathbf{i} + y\mathbf{j})}{2\sqrt{x^2 + y^2}} \\ &= \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{16}} \\ &= \frac{x\mathbf{i} + y\mathbf{j}}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{F} \cdot \mathbf{N} &= z \cdot \left(\frac{x}{4}\right) + xy \cdot \left(\frac{y}{4}\right) \\ &= \frac{zx + xy}{4} \\ &= \frac{x(y+z)}{4} \end{aligned}$$

For the limits of y , put

$$x^2 + y^2 = 16 \quad (\text{in } YZ \text{ plane } x=0)$$

$$0 + y^2 = 16$$

$$y = \pm 4$$

but surface is included in 1st octant

$$y = 0 \text{ to } y = 4$$

$$\iint_S \mathbf{F} \cdot \mathbf{N} \cdot d\mathbf{s} = \iint_S \mathbf{F} \cdot \mathbf{N} \cdot \frac{dy \cdot dz}{|\mathbf{N} \cdot \mathbf{i}|}$$

$$= \int_{y=0}^4 \int_{z=0}^5 \left(\frac{x}{4} \right) (y \cdot 1 + z \cdot 1) \cdot \frac{dy \cdot dz}{(x/4)}$$

$$= \left(\frac{y^2}{2} \right)_0^4 \cdot (z)_0^5 + \left(\frac{z^2}{2} \right)_0^5 (y)_0^4$$

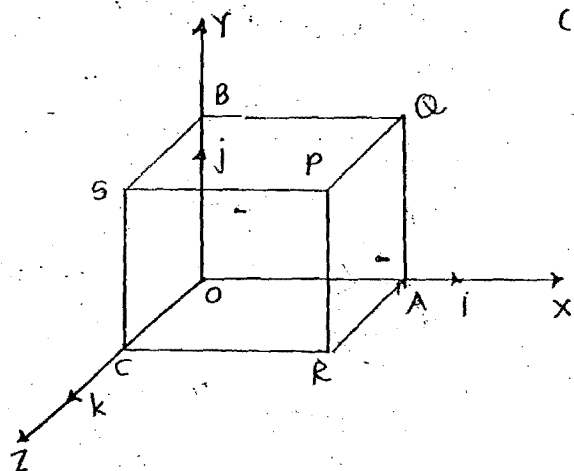
$$= \frac{16}{2} \times 5 + \frac{25}{2} \times 4^2$$

$$= 40 + 50$$

$$= 90$$

Friday
27th September 2013

Q. Find $\iint_S \mathbf{F} \cdot \mathbf{N} \cdot d\mathbf{s}$ where $\mathbf{F} = 4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$ and S is cube bounded by $0 \leq x, y, z \leq 1$



(i) For PQAR

$$\mathbf{N} = \mathbf{i} \quad x = 1$$

$$\mathbf{F} \cdot \mathbf{N} = (4xz) \cdot 1$$

$$= 4xz$$

$$= 4z$$

$$(\because x=1)$$

$$= \int_0^1 \int_0^1 4z \cdot \frac{dy \cdot dz}{|\mathbf{N} \cdot \mathbf{i}|}$$

$$= \int_0^1 \int_0^1 4z \cdot \frac{dy \cdot dz}{|\mathbf{i} \cdot \mathbf{i}|}$$

$$= \int_0^1 \int_0^1 4z \cdot dy \cdot dz \quad \mathbf{i} \cdot \mathbf{i} = 1$$

$$= 4 \left(\frac{z^2}{2} \right)_0^1 (y)_0^1 = 4 \left(\frac{1}{2} \right) \times 1 = 2$$

(ii) For OCSB,

$$\mathbf{N} = -\mathbf{i}, \quad x = 0$$

$$\mathbf{F} \cdot \mathbf{N} = -4xz = 0$$

$$(\because x=0)$$

$$\iint \mathbf{F} \cdot \mathbf{N} \cdot d\mathbf{s} = 0$$

(ii) For PQBS

$$\vec{N} = -\vec{j} \quad y=1$$

$$\vec{F} \cdot \vec{N} = -y^2 = -1$$

$$(\because y=1)$$

(67)

$$\begin{aligned} \iint \vec{F} \cdot \vec{N} \cdot d\vec{s} &= \int_0^1 \int_0^1 -1 \cdot dx \cdot dz \\ &= - \left[x \right]_0^1 \left[z \right]_0^1 \\ &= -1 \end{aligned}$$

(iv) For OARC

$$\vec{N} = -\vec{j} \quad y=0$$

$$\vec{F} \cdot \vec{N} = 0$$

$$\iint \vec{F} \cdot \vec{N} \cdot d\vec{s} = 0$$

(v) For PRCS

$$\vec{N} = \vec{k} \quad z=1$$

$$\vec{F} \cdot \vec{N} = yz = y \quad (\because z=1)$$

$$\begin{aligned} \iint \vec{F} \cdot \vec{N} \cdot d\vec{s} &= \int_0^1 \int_0^1 y^2 \cdot dx \cdot dz \\ &= \left[\frac{y^2}{2} \right]_0^1 \left[x \right]_0^1 \\ &= \frac{1}{2} \end{aligned}$$

(vi) For AOB

$$\vec{N} = -\vec{k} \quad z=0$$

$$\vec{F} \cdot \vec{N} = 0$$

$$\iint \vec{F} \cdot \vec{N} \cdot d\vec{s} = 0$$

$$\begin{aligned} \iint_S \vec{F} \cdot \vec{N} \cdot d\vec{s} &= 2 + 0 + (-1) + 0 + \left(\frac{1}{2}\right) \\ &= \frac{3}{2} \end{aligned}$$

(3) Volume integrals:

Any integral which is evaluated over a volume is called volume integral.

Q. Evaluate $\iiint_V \nabla \cdot F \cdot dv$ where $F = (2x^2 - 3z)i - 2xyj - 4zk$ and V is the region bounded by $x=0$, $y=0$, $z=0$ and $2x+2y+z=4$.

$$\nabla F = \frac{\partial F_1}{\partial x} i + \frac{\partial F_2}{\partial y} j + \frac{\partial F_3}{\partial z}$$

$$= 4x - 2x + 0$$

$$= 2x$$

Limits

$$\begin{aligned} z=0 & \text{ to } z=4-2x-2y \\ y=0 & \text{ to } y=2-x \\ x=0 & \text{ to } x=2 \end{aligned}$$

$$\iiint_V \nabla \cdot F \cdot dv = \int_{x=0}^2 \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 2x \cdot dx \cdot dy \cdot dz$$

$$= 2 \int_{x=0}^2 \int_{y=0}^{2-x} x (4-2x-2y) dx \cdot dy$$

$$= 2 \int_{x=0}^2 x \int_{y=0}^{2-x} 2(2-x)-2y \cdot dy \cdot dx$$

$$= 2 \int_{x=0}^2 x \left[2(2-x) \cdot y - 2 \frac{y^2}{2} \right]_0^{2-x} dx$$

$$= 2 \int_0^2 x \left[2(2-x)(2-x) - \frac{2(2-x)^2}{2} \right] dx$$

$$= 2 \int_0^2 x [2-x]^2 dx$$

$$= 2 \int_0^2 4x + x^3 - 4x^2 dx$$

$$= 2 \left[\frac{4x^2}{2} + \frac{x^4}{4} - \frac{4x^3}{3} \right]_0^2$$

$$= 2 \left[2 \times 8 + \frac{16}{4} - \frac{4 \times 8}{3} \right]$$

$$= \frac{8}{3}$$

(68)

Vector transformation:

1. Gauss-Divergence theorem ($\iint \rightarrow \iiint$)

Let S be the closed surface enclosed by a volume V .
Let F be any differentiable vector function, then,

$$\iint_S F \cdot N \cdot ds = \iiint_V (\text{div. } F) \cdot dv \quad (\text{only for closed surfaces})$$

or

$$\iint_S (F_1 \cdot dy \cdot dz + F_2 \cdot dz \cdot dx + F_3 \cdot dx \cdot dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx \cdot dy \cdot dz$$

Q. $\iint_S \vec{r} \cdot \vec{N} \cdot ds$ where S is closed surface enclosed by volume V

(a) V

(b) $2V$

(c) $3V$

(d) $4V$

$$\iint_S \vec{r} \cdot \vec{N} \cdot ds = \iiint_V \nabla \cdot \vec{r} \cdot dv$$

$$\int_V \vec{r} \cdot \vec{N} \cdot ds = \int_V (\nabla \cdot \vec{r}) \cdot dv +$$

$$= \int_V 3 \cdot dv$$

$$= 3V$$

$$\int_S 5 \cdot \vec{r} \cdot \vec{N} \cdot ds = 15V$$

$\int_S \vec{r} \cdot \vec{N} \cdot d\vec{s}$ where s is sphere $x^2 + y^2 + z^2 = a^2$

$$\begin{aligned} \int_S \vec{r} \cdot \vec{N} \cdot d\vec{s} &= \int_V \vec{r} \cdot d\vec{v} \\ &= 3V \\ &= 3 \times \frac{4}{3} \pi a^3 \\ &= 4\pi a^3 \end{aligned}$$

For radius R

$$\int_S \vec{r} \cdot \vec{N} \cdot d\vec{s} = 4\pi R^3$$

For $x^2 + y^2 + z^2 = 1$

$$\int_S \vec{r} \cdot \vec{N} \cdot d\vec{s} = 4\pi$$

$\int_S (x \cdot dy \cdot dz + y \cdot dz \cdot dx + z \cdot dx \cdot dy)$ where s is cube of unit length

$$\begin{aligned} \nabla F &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \int_S F \cdot \vec{N} \cdot d\vec{s} &= \int_V 3 \cdot d\vec{v} \\ &= 3V \\ &= 3(1) = 3 \end{aligned}$$

(unit length)

Find $\int_S (x+z) \cdot dy \cdot dz + (y+z) \cdot dz \cdot dx + (x+y) \cdot dx \cdot dy$ where s is the sphere $x^2 + y^2 + z^2 = 4$

$$\begin{aligned} \nabla \cdot F &= 1 + 1 + 0 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \int_S F \cdot \vec{N} \cdot d\vec{s} &= \int_V 2 \cdot d\vec{v} \\ &= 2V = 2 \times \frac{4}{3} \pi \times 2^3 \\ &= \frac{64\pi}{3} \end{aligned}$$

Find value of $\int_S (axi + byj + czk) \cdot N \, ds$ where S is sphere
of $x^2 + y^2 + z^2 = 1$ (6)

$$\nabla F = a + b + c$$

$$\begin{aligned} \therefore \int_S F \cdot N \, ds &= \int_V (a + b + c) \, dv \\ &= (a + b + c) \cdot V \\ &= (a + b + c) \times \frac{4}{3} \times \pi \times 1 \\ &= \frac{4}{3} \pi (a + b + c) \end{aligned}$$

Find $\int_S (ax^2 + by^2 + cz^2) \cdot ds$ where S is $x^2 + y^2 + z^2 = 1$

$$N = \frac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}}$$

$$= xi + yj + zk$$

$$\therefore \text{for } Caxi + byj + czk = F.$$

$$F \cdot N = ax^2 + by^2 + cz^2$$

$$\begin{aligned} \int_S (ax^2 + by^2 + cz^2) \cdot ds &= \int_S (Caxi + byj + czk) \cdot N \cdot ds. \\ &= \frac{4}{3} \pi (a + b + c) \end{aligned}$$

Find $\int_S (4xz i - y^2 j + yz k) \cdot N \, ds$ where S is the cube bounded
by $0 \leq x, y, z \leq 1$

$$\begin{aligned} \nabla F &= 4z - 2y + 0 \\ &= 4z - y \end{aligned}$$

$$\int_S \mathbf{F} \cdot \mathbf{N} \cdot d\mathbf{s} = \int_V \nabla \mathbf{F} \cdot d\mathbf{V}$$

$$= \int_{x=0}^1 \int_{y=0}^1 \int_{z=0}^1 (4z - y) \cdot dx \cdot dy \cdot dz$$

$$= (x)_0^1 \left[4 \left(\frac{z^2}{2} \right)_0^1 (y)_0^1 - \left(\frac{y^2}{2} \right)_0^1 (z)_0^1 \right]$$

$$= \frac{3}{2}$$

Find $\int_S \mathbf{F} \cdot \mathbf{N} \cdot d\mathbf{s}$ where $\mathbf{F} = (x^2 - yz)\mathbf{i} + (y^2 - 2xz)\mathbf{j} + (z^2 - xy)\mathbf{k}$

where S is cuboid bounded by $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$

$$\nabla \mathbf{F} = 2x + 2y + 2z$$

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$$\therefore \int_S \mathbf{F} \cdot \mathbf{N} \cdot d\mathbf{s} = \int_{x=0}^a \int_{y=0}^b \int_{z=0}^c [2x \cdot \mathbf{i} + 2y \cdot \mathbf{j} + 2z \cdot \mathbf{k}] \cdot dx \cdot dy \cdot dz$$

$$= \left[2 \cdot \frac{x^2}{2} \cdot y \cdot z + 2 \cdot \frac{y^2}{2} \cdot x \cdot z + 2 \cdot \frac{z^2}{2} \cdot x \cdot y \right]_{x=0, y=0, z=0}^{x=a, y=b, z=c}$$

$$= (a^2bc + b^2ac + c^2ab)$$

$$= abc(a + b + c)$$

A region of volume 10 cu. units is bounded by closed surface and $\mathbf{N}$ is unit normal vector to the surface.

$$\iiint_S (4x\mathbf{i} + 2y\mathbf{j} - z\mathbf{k}) \cdot \mathbf{N} \cdot d\mathbf{s}$$

$$\iiint_V \nabla \mathbf{F} \cdot d\mathbf{V} = 10 \quad \therefore \nabla \mathbf{F} = 4 + 2 - 1 = 5$$

$$\iiint_S (4x\mathbf{i} + 2y\mathbf{j} - z\mathbf{k}) \cdot \mathbf{N} \cdot d\mathbf{s} = 5$$

2. Green's theorem in a plane ($\oint \rightarrow \iint$)

(90)

Let R be closed region in XY plane bounded by a simple and closed curve C . Let P and Q are continuous functions of x and y possessing the first order partial derivatives then

$$\oint_C P \cdot dx + Q \cdot dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cdot dx \cdot dy$$

- Selecting limits is important

- Draw diagram to understand limits

TATEOTOHAYO

- circle, square, rectangle - symmetrical - no need of diagram.

Find $\oint_C (x^2 + y^2) \cdot dx + (-2xy) \cdot dy$ where C is rectangle bounded by $x=0, x=a$ and $y=0, y=b$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} (x^2 + y^2) = 2y$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} (-2xy) = -2y$$

$$\oint_C (x^2 + y^2) \cdot dx - 2xy \cdot dy = \int_{x=0}^a \int_{y=0}^b (-2y - 2y) \cdot dx \cdot dy$$

$$= (x)_0^a \cdot \left(-4 \cdot \frac{y^2}{2} \right)_0^b$$

$$= a \left(-4 \times \frac{b^2}{2} \right)$$

$$= -2ab^2$$

Find $\oint_C (2x - y) \cdot dx + (x + y) \cdot dy$ where C is circle $x^2 + y^2 = 9$

$$\frac{\partial P}{\partial y} = -1 \quad \frac{\partial Q}{\partial x} = 1$$

$$= \iint_R [1 - (-1)] \cdot dx \cdot dy$$

$$= \int_R 2 \cdot dR$$

$$= 2 \cdot R$$

$$= 2 \times \pi (9)$$

$$= 18\pi$$

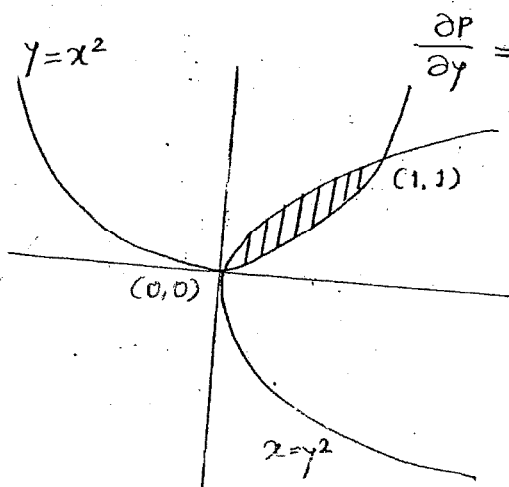
Find $\oint_C (x^2 - xy^3) \cdot dx + (y^3 - 2xy) \cdot dy$ where C is rectangle bdd by $(0,0), (2,0), (2,2), (0,2)$

$$\frac{\partial P}{\partial y} = -3xy^2$$

$$\frac{\partial Q}{\partial x} = -2y$$

$$\begin{aligned} \oint_C P dx + Q \cdot dy &= \int_0^2 \int_0^2 (+3xy^2 - 2y) \cdot dx \cdot dy \\ &= \left(-\frac{2y^2}{2} \right)_0^2 \cdot (x)_0^2 + 3 \left(\frac{x^2}{2} \right)_0^2 \cdot \left(\frac{y^3}{3} \right)_0^2 \\ &= -8 + 16 \\ &= 8 \end{aligned}$$

Find $\oint_C (3x^2 - 8y^2) \cdot dx + (4y - 6xy) \cdot dy$ where C is region bdd by $y = \sqrt{x}$ and $y = x^2$.



$$\frac{\partial P}{\partial y} = -16y$$

$$\frac{\partial Q}{\partial x} = -6y$$

Functions limits required.

$$y = \sqrt{x} \Rightarrow x = y^2$$

$$\text{and } y = x^2$$

$$x^2 = \sqrt{x}$$

$$x^4 = x$$

$$x^4 - x = 0$$

$$x(x^3-1)=0$$

$$x=0, 1$$

$$y=0, 1$$

point of intersection $(0,0)$ and $(1,1)$ -

If two curves bound a region and region is not symmetric about any axis, then lower curve represent lower limits and upper curve given upper limit.

$\therefore$ limits of $y = x^2$ to $\sqrt{x}$

$x = 0$ to 1

$$\int_{x=0}^1 \int_{y=x^2}^{\sqrt{x}} (-6y+16y) \cdot dx \cdot dy$$

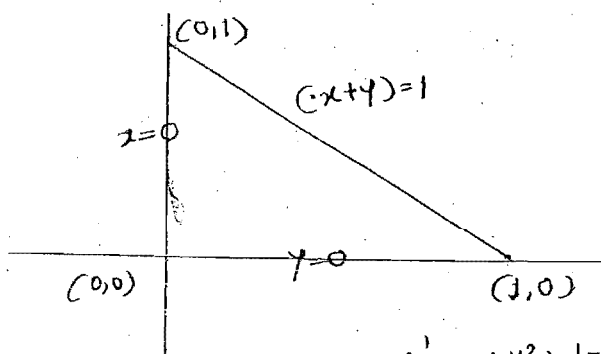
$$= \int_0^1 \left(10 \cdot \frac{y^2}{2} \right)_{x^2}^{\sqrt{x}} dx$$

$$= 5 \int_0^1 (x - x^4) \cdot dx$$

$$= \frac{3}{2}$$

Find the $\oint Pdx + Q \cdot dy$ of above problem for region bdd by

$$x=0, y=0, x+y=1$$



Limits of y

$$y=0 \text{ to } y=(1-x)$$

limits of x

$$x=0 \text{ to } x=1$$

$$= \int_{x=0}^1 \int_0^{1-x} (-6y+16y) \cdot dx \cdot dy$$

$$= \int_0^1 10 \left(\frac{y^2}{2} \right)_0^{1-x} dx = 5 \int_0^1 (1-x)^2 dx = \frac{5}{3}$$

Find $\oint_C (y - \sin x) \cdot dx + \cos x \cdot dy$ where C is Δ -formed by
 $y=0, x=\frac{\pi}{2}, y=\frac{2x}{\pi}$ Ans. $-\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$

Find $\oint_C (x^2 - \cosh y) \cdot dx + (y + \sin x) \cdot dy$ where C is rectangle bdd
 by $(0,0) (\pi,0) (\pi,1) (0,1)$. Ans. $\pi [\cosh 1 - 1]$

3. Stoke's theorem ($\oint \rightarrow \iint$)

Let 'S' be an open surface bounded by a closed and non-intersecting curve C . Let F be any differentiable vector function then.

$$\begin{aligned} \oint_C F \cdot d\vec{r} &= \iint_S \text{Curl } F \cdot \vec{N} \cdot dS \\ &= \iint_S (\nabla \times F) \cdot \vec{N} \cdot dS \end{aligned}$$

Evaluate $\oint_C F \cdot d\vec{r}$, if $F = -y^3 \vec{i} + x^3 \vec{j}$ and S is surface of the circular disc $x^2 + y^2 \leq 1, z=0$. (can solve by Green's theorem)

$$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= \vec{i}(-0) - \vec{j}(0-0) + \vec{k}(3x^2 + 3y^2)$$

$$= 3(x^2 + y^2) \cdot \vec{k}$$

$$= 3\vec{k}$$

$$\therefore x^2 + y^2 \leq 1$$

$$\therefore \oint_C F \cdot d\vec{r} = \iint_S (\nabla \times F) \cdot \vec{N} \cdot dS$$

$$= \int_S 3\vec{k} \cdot \vec{k} \cdot dS$$

$\swarrow$ $z=0$ $\therefore xy$ plane

$$= \int 3k^2 \cdot dS$$

$$= \int_0^5 3 \cdot ds$$

$$\therefore k^2 = 1$$

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$$= 3 \cdot 5$$

$$= 3 \times \pi (1)^2$$

$$= 3\pi$$

If $y = x + \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}}$ $y(2) = ?$

(a) 4 or 1

(b) 1 only

(c) 4 only

(d) undefined.

$$y = x + \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}$$

$$y - x = \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}$$

$$y - x = \sqrt{x + (y - x)}$$

$$(y - x)^2 = x + y - x$$

$$(y - 1)(y - 4) = 0$$

$$y = 1 \text{ or } y = 4$$

but value of y should be more than 2 here for $y(2)$

Find $\lim_{x \rightarrow 0} \frac{e^x - (1 + x + \frac{x^2}{2!})}{x^3} =$

(a) $\frac{1}{6}$

(b) $\frac{1}{3}$

(c) $\frac{1}{2}$

(d) 1

$$L = \lim_{x \rightarrow 0} \frac{(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots) - (1 + x + \frac{x^2}{2!})}{x^3}$$

$$= \frac{1}{6} + 0 + 0$$

$$= \frac{1}{6}$$

$$\text{If } f(x) = \frac{2x^2 - 7x + 3}{5x^2 - 12x - 9}$$

$$\therefore \lim_{x \rightarrow 3} f(x) = ?$$

$$(a) -1/3$$

$$(b) 0$$

$$(c) \frac{2}{5}$$

$$(d) \frac{5}{18}$$

By L-hospitality rule

$$\begin{aligned} \lim_{x \rightarrow 3} f(x) &= \frac{4x - 7}{10x - 12} \\ &= \frac{5}{18} \end{aligned}$$

$$\text{Find } \int_0^{\pi/3} e^{it} \cdot dt$$

$$I = \int_0^{\pi/3} e^{it} \cdot dt$$

$$\therefore e^{i\theta} = \cos \theta + i \sin \theta$$

$$= \int_0^{\pi/3} (\cos t + i \sin t) \cdot dt$$

$$= [\sin t - i \cdot \cos t]_0^{\pi/3}$$

$$= \left[\sin \frac{\pi}{3} - i \cos \frac{\pi}{3} \right] - [\sin 0 - i \cos 0]$$

$$= \frac{\sqrt{3}}{2} - \frac{i}{2} + i$$

$$= \frac{\sqrt{3}}{2} + \frac{i}{2}$$

With $y = e^{ax}$ if the sum $S = \frac{dy}{dx} + \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} + \dots + \frac{d^ny}{dx^n}$ approaches to $2y$ as $n \rightarrow \infty$. Find a .

$$S = 2y$$

$$2y = \frac{d}{dx}(e^{ax}) + \frac{d^2}{dx^2}(e^{ax}) + \dots$$

$$2 \cdot e^{ax} = a \cdot e^{ax} + a^2 \cdot e^{ax} + a^3 \cdot e^{ax} + \dots$$

$$2 = a + a^2 + a^3 + \dots$$

$\therefore$ Sum of infinite Geometric progression $= S_{\infty} = \frac{a}{1-r}$

$$r = \frac{T_2}{T_1}$$

$$= \frac{a^2}{a} = a.$$

$$2 = \frac{a}{1-a}$$

$$a = \frac{2}{3}$$

If two roots of non-linear equation $f(x) = x^3 - 6x^2 + 11x - 6 = 0$ are 1 and 3 then third root is.

sum of three roots

$$\alpha + \beta + \gamma = \frac{-b}{a}$$

$$= \frac{-(-6)}{1}$$

$$3 + 1 + \gamma = 6$$

$$\gamma = 2$$

or

1st value always zero

	1	-6	11	-6
$x=1$	0	1	-5	6
	1	-5	6	0
$x=3$	0	3	-6	
	1	-2	0	0

Method of the synthetic division.

$$x - 2 = 0$$

$$x = 2$$

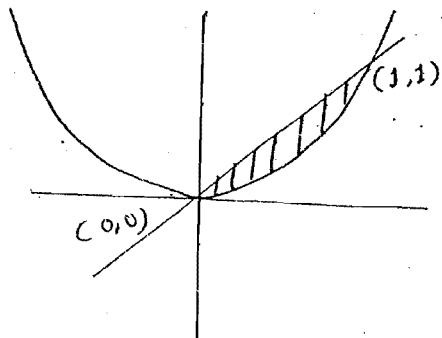
The area of the region bounded by $y = x^2$ and $y = x$

(a) $\frac{1}{6}$

(b) $\frac{1}{3}$

(c) $\frac{1}{2}$

(d) 1



$$x^2 = x$$

$$x(x-1) = 0$$

$$x = 0 \quad x = 1$$

$$\downarrow \quad \downarrow$$

$$y = 0 \quad y = 1$$

$$(0,0) \quad (1,1)$$

$$\begin{aligned}
 \text{Area} &= \int_0^1 (x - x^2) \cdot dx \quad (\text{upper curve} - \text{lower curve}) \\
 &= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 \\
 &= \frac{1}{2} - \frac{1}{3} \\
 &= \frac{1}{6}
 \end{aligned}$$

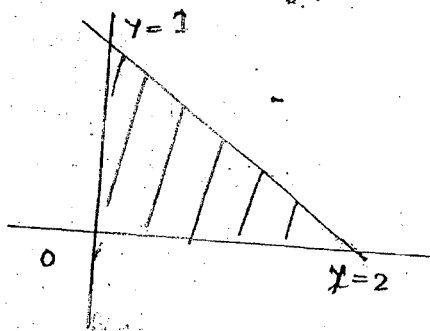
Area bounded by $y^2 = 4ax$ & $x^2 = 4ay \Rightarrow \frac{16a^2}{3}$

$y^2 = 8x$ & $x^2 = 8y \Rightarrow \frac{16(2)^2}{3} = \frac{64}{3}$

$y^2 = 4x$ & $x^2 = 4y \Rightarrow \frac{16}{3} \quad (a=1)$

$y^2 = x$ & $x^2 = y \Rightarrow \frac{1}{3} \quad a = \frac{1}{4}$

∬ xy · dx · dy for given region is



Equⁿ of line
 $= \frac{x}{2} + \frac{y}{1} = 1$

∴ limits $y = 0$ to $1 - \frac{x}{2}$
 $x = 0$ to 2

$$\iint xy \cdot dx \cdot dy = \int_{x=0}^2 \int_{y=0}^{1-x/2} xy \cdot dx \cdot dy$$

$$= \int_{x=0}^2 x \left[\frac{y^2}{2} \right]_0^{1-x/2} dx$$

$$= \int_0^2 \frac{x}{2} \left[\left(1 - \frac{x}{2} \right)^2 \right] dx$$

$$= \int_0^2 \frac{x}{2} \left[1 - x + \frac{x^2}{4} \right] dx$$

$$= \frac{1}{6}$$

$$\int_0^3 \int_0^x (6-x-y) \cdot dx \cdot dy$$

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(a) 13.5

(b) 27.0

(c) 40.5

(d) 54.0

$$J = \int_0^3 \left(6y - xy - \frac{y^2}{2} \right)_0^x dx$$

$$= \int_0^3 \left(6x - x^2 - \frac{x^2}{2} \right) dx$$

$$= \int_0^3 \left(6x - \frac{3x^2}{2} \right) dx$$

$$= 6 \left(\frac{x^2}{2} \right)_0^3 - \frac{3}{2} \left(\frac{x^3}{3} \right)_0^3$$

$$= 3 \times 9 - \frac{1}{2} (27)$$

$$= 27 - 12.5$$

$$= 13.5$$

A parabolic arc $y = \sqrt{x}$ $1 \leq x \leq 2$ is revolved around x -axis then the volume of solid revolution is .

$$\text{volume} = \int_a^b \pi y^2 dx$$

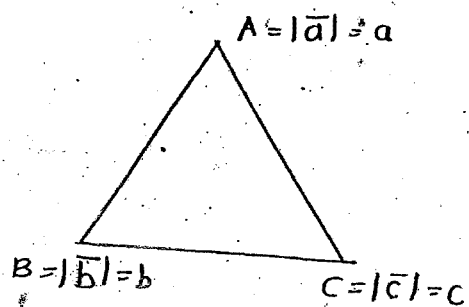
$$= \int_1^2 \pi \cdot x \cdot dx$$

$$= \pi \left[\frac{x^2}{2} \right]_1^2$$

$$= \frac{\pi}{2} (4-1)$$

$$= \frac{3\pi}{2}$$

The area of Δ formed by tips of vector $\vec{a}$, $\vec{b}$ & $\vec{c}$ is



$$\begin{aligned} \text{Area} &= \frac{1}{2} |\vec{AB} \times \vec{AC}| \\ &= \frac{1}{2} |(\vec{B} - \vec{A}) \times (\vec{C} - \vec{A})| \\ &= \frac{1}{2} |(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})| \\ &= \frac{1}{2} |(\vec{a} - \vec{b}) \times (\vec{a} - \vec{c})| \end{aligned}$$

The value of α for which the vectors are co-planer.

$$\vec{a} = \vec{i} + 2\vec{j} + \vec{k}$$

$$\vec{b} = 3\vec{i} + \vec{k}$$

$$\vec{c} = 2\vec{i} + \alpha\vec{j}$$

vectors are co-planer

$$[\vec{a} \ \vec{b} \ \vec{c}] = 0$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & \alpha & 0 \end{vmatrix} = 0$$

$$(0 - \alpha) - 2(0 - 2) + 1(3\alpha - 0) = 0$$

$$2\alpha - 4 = 0$$

$$\alpha = 2$$

Differential equations:

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Any equation consisting of differential coefficients

$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}$ is called Differential equation.

Degree of differential equation:

The degree of differential eqn is degree of highest order.

Find the degree and order of following D.E.

1. $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0$

degree = 1

order = 2

2. $\left(\frac{d^2y}{dx^2}\right)^3 + 5\frac{dy}{dx} + 6y = k\left(\frac{d^3y}{dx^3}\right)^2$

degree = 2

order = 3

3. $\left(\frac{d^2y}{dx^2}\right)^2 + \sqrt{1 + \left(\frac{dy}{dx}\right)^3} = 5y$

degree = 4

order = 2

4. $\left(\frac{d^2y}{dx^2}\right) + 6 \cos\left(\frac{dy}{dx}\right) + 7y = 6$

degree - not defined

order = 2

Formation of differential equation :

1. differential equation can be formed by eliminating arbitrary constants in the solution.

$$y^2 = 4ax$$

constant (doesn't change) arbitrary constant (can change)

$$2y \cdot \frac{dy}{dx} = 4a$$

$$(1). \quad y^2 = x \cdot 2y \cdot \frac{dy}{dx}$$

$$y = 2x \cdot \frac{dy}{dx}$$

$$2x \cdot \frac{dy}{dx} - y = 0$$

$$2. \quad x^2 + y^2 = a^2$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$x + y \cdot \frac{dy}{dx} = 0$$

$$3. \quad y^2 = 4a(x+a)$$

$$2y \cdot \frac{dy}{dx} = 4a \quad (1)$$

$$y \cdot \frac{dy}{dx} = 2a$$

$$y \cdot y_1 = 2a$$

$$a = \frac{y \cdot y_1}{2}$$

$$y^2 = \frac{4 \cdot y \cdot y_1}{2} \cdot \left(x + \frac{y \cdot y_1}{2}\right)$$

$$y = 2 \cdot \frac{dy}{dx} \cdot \left(x + y \cdot \frac{dy}{2 dx}\right)$$

$$y \cdot \left(\frac{dy}{dx}\right)^2 - 2x \cdot \frac{dy}{dx} + y = 0$$

$$y = a \cdot e^x + b \cdot e^{-x}$$

$$\frac{dy}{dx} = a \cdot e^x + b \cdot e^{-x} (-1)$$

$$\frac{dy}{dx} = a \cdot e^x - b \cdot e^{-x}$$

$$\frac{dy}{dx} = - (a \cdot e^x + b \cdot e^{-x})$$

$$\frac{dy}{dx} + y = 0$$

or

$$\frac{d^2y}{dx^2} - y = 0$$

$$y = e^x (a \cos x + b \sin x)$$

$$\frac{dy}{dx} = e^x (-a \sin x + b \cos x) + e^x (a \cos x + b \sin x)$$

$$= e^x (a \cos x + b \sin x) + e^x (-a \sin x + b \cos x)$$

$$y_1 = y + e^x (-a \sin x + b \cos x)$$

$$y_1 - y = e^x (-a \sin x + b \cos x)$$

$$y_2 - y_1 = e^x (-a \cos x - b \sin x) + (-a \sin x + b \cos x)$$

$$y_2 - y_1 = -e^x (a \cos x + b \sin x) + (-a \sin x + b \cos x)$$

$$y_2 - y_1 = -y + (y_1 - y)$$

$$\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$$

Solution of differential equation:

(1) Variable-separable method:

$$1. \frac{dy}{dx} + \sqrt{\frac{1+y^2}{1+x^2}} = 0$$

$$\frac{dy}{dx} = \frac{-\sqrt{1-y^2}}{\sqrt{1+x^2}}$$

$$\frac{dy}{\sqrt{1-y^2}} = \frac{-dx}{\sqrt{1+x^2}}$$

$$\int \frac{dy}{\sqrt{1-y^2}} = - \int \frac{dx}{\sqrt{1+x^2}}$$

$$\sinh^{-1} y = - \sinh^{-1} x + c$$

$$\sinh^{-1} x + \sinh^{-1} y = c$$

$$\frac{dy}{dx} = e^{x+y}$$

$$\frac{dy}{dx} = e^x \cdot e^y$$

$$\frac{dy}{e^y} = e^x dx$$

$$\int e^{-y} dy = \int e^x dx$$

$$\frac{e^{-y}}{-1} = e^x + c$$

$$e^x + e^{-y} + c = 0$$

$$e^x \cdot e^y + 1 + c \cdot e^y = 0$$

$$e^{x+y} + c \cdot e^y + 1 = 0$$

$$\frac{dy}{dx} = x^2 \cdot e^{x-y}$$

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$$\frac{dy}{dx} = x^2 \cdot \frac{e^x}{e^y}$$

$$\int e^y dy = \int x^2 \cdot e^x \cdot dx$$

$$e^y = e^x (x^2 - 2x + 2) + c$$

$$\frac{dy}{dx} + y^2 = 0$$

$$\frac{dy}{dx} = -y^2$$

$$\int \frac{-1}{y^2} dy = \int dx$$

$$\frac{y^{-1}}{-1} = x + c$$

$$x + \frac{1}{y} + c = 0$$

$$y = \frac{1}{x+c}$$

$$\frac{dy}{dx} + y^3 = 0 \quad y(1) = 1, \text{ Find the interval of validity.}$$

$$\frac{dy}{dx} = -y^3$$

$$\int y^{-3} dy = - \int dx$$

$$\frac{y^{-2}}{-2} = -x + c$$

$$\frac{(1)^{-2}}{-2} = -1 + c$$

$$c = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$\frac{y^{-2}}{-2} = -\left(x - \frac{1}{2}\right)$$

$$\frac{1}{2y^2} = x - \frac{1}{2}$$

$$y^2 = \frac{1}{2(x - \frac{1}{2})}$$

For validity

$$(x - \frac{1}{2}) > 0$$

$$x > \frac{1}{2}$$

∵ Square of number always +ve

∴ Interval of validity, $\frac{1}{2} < x < \infty$

$$e^x \tan y \cdot dx + (1 + e^x) \cdot \sec^2 y \cdot dy = 0$$

$$e^x \tan y \cdot dx = - (1 + e^x) \cdot \sec^2 y \cdot dy$$

$$\frac{e^x}{1 + e^x} dx = - \frac{\sec^2 y}{\tan y} dy$$

$$\int \frac{e^x}{1 + e^x} dx + \int \frac{\sec^2 y}{\tan y} dy = 0$$

$$\log(1 + e^x) + \log(\tan y) = \log c$$

$$(1 + e^x) \cdot \tan y = c$$

(2) Equation reducible to variable-separable form:

If given D.E. is not directly under variable-separable form, by making substitution again it can be transformed in variable-separable form.

$$\frac{dy}{dx} = (4x + y + 1)^2$$

$$Z = 4x + y + 1$$

$$\frac{dz}{dx} = 4 + \frac{dy}{dx}$$

$$\frac{dz}{dx} = 4 + z^2$$

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$$\int \frac{dz}{z^2 + z^2} = \int dx$$

$$\frac{1}{2} \cdot \tan^{-1}\left(\frac{z}{2}\right) = x + c$$

$$\frac{1}{2} \tan^{-1}\left(\frac{4x+y+1}{2}\right) - x = c$$

$$\frac{dy}{dx} = \frac{a^2}{(x-y)^2}$$

$$x - y = z$$

$$1 - \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dy}{dx} = 1 - \frac{dz}{dx}$$

$$\frac{a^2}{z^2} = 1 - \frac{dz}{dx}$$

$$\frac{dz}{dx} = \frac{z^2 - a^2}{z^2}$$

$$\int \frac{z^2}{z^2 - a^2} \cdot dz = \int dx$$

$$z^2 - a^2 \left[\frac{z^2}{-z^2 - a} \right] \frac{1}{a}$$

$$z + a^2 \cdot \frac{1}{2a} \log\left(\frac{z-a}{z+a}\right) = x + c$$

or

$$x - y + a^2 \cdot \frac{1}{2a} \cdot \log\left(\frac{x-y-a}{x-y+a}\right) = x + c$$

$$\frac{z^2 - a^2 + a^2}{z^2 - a^2}$$

$$y + c = \frac{a}{2} \log\left(\frac{x-y-a}{x-y+a}\right) \cdot 1 + \frac{a^2}{(z^2 - a^2)}$$

43) Homogenous differential equation :

A D.E. is of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is said to be

a homogenous differential equation if both $f(x,y)$ and $g(x,y)$ are homogenous functions of same degree.

Here substitute,

$$y = vx \Rightarrow v = \frac{y}{x}$$

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

or

$$x = vy \Rightarrow v = \frac{x}{y}$$

$$\frac{dx}{dy} = v \cdot \frac{dy}{dy} + y \cdot \frac{dv}{dy}$$

(For D.E. only algebraic, trigonometric & exponential function exist)

e.g.

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \quad \text{or} \quad \frac{dx}{dy} = \frac{2xy}{x^2 + y^2}$$

the solution will be same for both functions but "preference is given to the function which have more no. of terms in numerator than denominator (for algebraic function in both numerator and denominator)

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

put

$$y = vx$$

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

$$v + x \cdot \frac{dv}{dx} = \frac{x^2 + v^2 x^2}{2x \cdot (x^2 v^2)}$$

$$x \cdot \frac{dv}{dx} = \frac{1 + v^2}{2v} - v$$

$$\int \frac{1}{x} \cdot dx = \log cx$$

$$\int \frac{-1}{x} \cdot dx = \log \frac{c}{x}$$

$$\int \frac{2}{x} \cdot dx = \log cx^2$$

$$\int \frac{-3}{x} \cdot dx = \log \frac{c}{x^3}$$

$$\int \frac{-2}{x} \cdot dx = \log \frac{c}{x^2}$$

$$x \cdot \frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$$

$$= \frac{1-v^2}{2v}$$

$$\int \frac{-2v}{1-v^2} dv = \int \frac{-dx}{x}$$

$$\log(1-v^2) = \log \frac{c}{x}$$

$$1 - \frac{y^2}{x^2} = \frac{c}{x}$$

$$\frac{x^2 - y^2}{x^2} = \frac{c}{x}$$

$$x^2 - y^2 = c \cdot x$$

$$x^2 y \cdot dx - (x^3 + y^3) dy = 0$$

$$x^3 y \cdot dx = (x^3 + y^3) \cdot dy$$

$$\frac{dx}{dy} = \frac{(x^3 + y^3)}{x^3 y}$$

$$= \frac{x^3}{x^3 y} + \frac{y^3}{x^3 y}$$

$$\frac{dx}{dy} = \frac{x}{y} + \frac{y}{x}$$

$$v + y \cdot \frac{dv}{dy} = v + \frac{1}{v^2}$$

$$\int v^2 \cdot dv = \int \frac{dy}{y}$$

$$\frac{v^3}{3} = \log y + c$$

$$\frac{x^3}{3y^3} = \log y + c$$

$$x \cdot \frac{dy}{dx} = y + x \tan\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$$

$$v + x \frac{dv}{dx} = v + \tan v$$

$$\frac{dv}{\tan v} = \frac{dx}{x}$$

$$\int \cot v \cdot dv = \int \frac{dx}{x}$$

$$\log(\sin v) = \log c x$$

$$\sin\left(\frac{y}{x}\right) = cx$$

$$(1 + e^{xy}) dx + e^{xy} \left(1 - \frac{x}{y}\right) dy = 0$$

$$(1 + e^{xy}) dx = e^{xy} \left(\frac{x-y}{y}\right) dy$$

$$\frac{dx}{dy} = \frac{e^{xy} (x/y - 1)}{(1 + e^{xy})}$$

$$v + y \frac{dv}{dy} = \frac{e^v (v-1)}{(1 + e^v)}$$

$$x = vy$$

$$y \cdot \frac{dv}{dy} = \frac{v \cdot e^v (v-1)}{(1 + e^v)} - v$$

$$y \cdot \frac{dv}{dy} = \frac{ve^v - e^v - v - v \cdot e^v}{(1 + e^v)}$$

$$y \cdot \frac{dv}{dy} = \frac{-(e^v + v)}{(1 + e^v)}$$

$$\int \frac{1 + e^v}{v + e^v} dv = \int \frac{-dy}{y}$$

$$\log(v + e^v) = \log(c/y)$$

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$$\frac{x}{y} + e^{xy} = \frac{c}{y}$$

$$x + y \cdot e^{xy} = c$$

(1) Non-homogenous equation.

The D.E. is of form $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$ is said to be a non-homogenous differential equation. because of constants the equⁿ is not homogene

Case I: If $a_1b_2 - a_2b_1 \neq 0$ then,

substitute $x = X + h$, $y = Y + k$.

where h and k are constants to be determined.

Case II: If $a_1b_2 - a_2b_1 = 0$ then, some part of numerator and denominator which is equal, is considered as substitution.

Find the substitution that transforms the given D.E. from non-homogenous to homogenous form.

$$\frac{dy}{dx} = \frac{3x + y - 5}{2x + y - 2}$$

$$6 - 2 \neq 0$$

Case I:

h	k	1	h
3	1	-5	3
2	2	-2	2

$$\frac{h}{-2+10} = \frac{k}{-10+6} = \frac{1}{6-2}$$

$$\frac{h}{8} = \frac{k}{-4} = \frac{1}{4}$$

$$h = \frac{8}{4} = 2$$

$$k = \frac{-4}{4} = -1$$

$$x = X + 2$$

$$y = Y - 1$$

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$$

$$h \quad k \quad 1 \quad h$$

$$1 \quad 2 \quad -3 \quad 1$$

$$2 \quad 1 \quad -3 \quad 2$$

$$\frac{h}{-6+3} = \frac{k}{-6+3} = \frac{1}{1-4}$$

$$\frac{h}{-3} = \frac{k}{-3} = \frac{1}{-5}$$

$$h = k = 1$$

$$x = X + 1$$

$$y = Y + 1$$

$$\frac{dy}{dx} = \frac{x+2y+1}{2x+4y+1}$$

$$4 - 2(2) = 0$$

$$\frac{dy}{dx} = \frac{(x+2y)+1}{2(x+2y)+1}$$

substitute

$$(x+2y) = z$$

$$\frac{dy}{dx} = \frac{x+2y+3}{2x+y+5}$$

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$$1-4 \neq 0$$

h	k	j	h
1	2	3	1
2	1	5	2

$$\frac{h}{10-3} = \frac{k}{6-5} = \frac{1}{1-4}$$

$$\frac{h}{7} = \frac{k}{1} = \frac{1}{-3}$$

$$h = -\frac{7}{3} \quad k = -\frac{1}{3}$$

$$x = X + \frac{7}{3}$$

$$y = Y - \frac{1}{3}$$

Exact differential equation:

A differential equation is of the form $M \cdot dx + N \cdot dy = 0$ is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

General solution is

$$\int_0^x M \cdot dx + \int (\text{terms of } N \text{ not containing } x) \cdot dy = c$$

$$(x^2+y^2) \cdot dx + 2xy \cdot dy = 0$$

$$\frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial x} = 2y \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

∴ solution is

$$\int_0^x (x^2+y^2) \cdot dx + \int 0 \cdot dy = c$$

$$\frac{x^3}{3} + xy^2 = c$$

$$2xy \cdot dx + (x^2 + y^2) \cdot dy = 0$$

$$\frac{\partial M}{\partial y} = 2x$$

$$\frac{\partial M}{\partial x} = 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial M}{\partial x}$$

$$\int_0^x 2xy \cdot dx + \int y^2 \cdot dy = c$$

$$x^2 y + \frac{y^3}{3} = c$$

shortcut :

$$(ax + hy + g) dx + (hx + by + f) dy = 0$$

$$\frac{\partial M}{\partial y} = h$$

$$\frac{\partial M}{\partial x} = h$$

∴ solution is

$$\frac{ax^2}{2} + hy(x) + gx + \frac{by^2}{2} + fy = c$$

$$ax^2 + 2hyx + 2gx + by^2 + 2fy = 2c$$

$$e^x \cdot \tan y \cdot dx + (1 + e^x) \cdot \sec^2 y \cdot dy = 0$$

$$\frac{\partial M}{\partial y} = e^x (\sec^2 y)$$

$$\frac{\partial M}{\partial x} = \sec^2 y (e^x)$$

$$\therefore \int e^x \cdot \tan y \cdot dx + \int \sec^2 y \cdot dy = c$$

$$\tan y \cdot e^x + \tan y = c$$

$$(e^x + 1) \tan y = c$$

$$(1 + e^{xy}) dx + e^{xy} (1 - \frac{x}{y}) dy = 0$$

$$\frac{\partial M}{\partial y} = e^{xy} \left(-\frac{x}{y^2} \right)$$

$$\frac{\partial M}{\partial x} = e^{xy} \left(-\frac{1}{y} \right) + \left(1 - \frac{x}{y} \right) e^{xy} \cdot \frac{1}{y}$$

$$= e^{xy} \left(-\frac{x}{y^2} \right)$$

$$\therefore \int (1 + e^{xy}) \cdot dx +$$

$$= x + y \cdot e^{xy} = c$$

Non-exact differential equation:

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The D.E. is of the form $M dx + N dy = 0$ is said to be non-exact if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Method 1:

To form suitable integrating factors:

(i) $x dy + y dx = d(xy)$

(ii) $\frac{x dy - y dx}{x^2} = d\left(\frac{y}{x}\right)$

(iii) $\frac{x dy - y dx}{xy} = d(\log y/x)$

(iv) $\frac{x dy - y dx}{x^2 + y^2} = d(\tan^{-1} y/x)$

(v) $\frac{y dx - x dy}{y^2} = d\left(\frac{x}{y}\right)$

(vi) $\frac{y dx - x dy}{xy} = d(\log \frac{x}{y})$

(vii) $\frac{y dx - x dy}{x^2 + y^2} = d(\tan^{-1}(x/y))$

(viii) $\frac{y \cdot e^x dx - e^x dy}{y^2} = d(e^x/y)$

(ix) $\frac{2xy dx - x^2 dy}{y^2} = d(x^2/y)$

(x) $\frac{x dx + y dy}{x^2 + y^2} = d\left(\frac{1}{2} \log(x^2 + y^2)\right)$

$$Q. y(1+xy) dx + x(1-xy) \cdot dy = 0$$

$$(y + xy^2) \cdot dx + (x - x^2y) \cdot dy = 0$$

$$\frac{\partial M}{\partial y} = 1 + 2xy$$

$$\frac{\partial N}{\partial x} = 1 - 2xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$y \cdot dx + xy^2 \cdot dx + x \cdot dy - x^2y \cdot dy$$

$$\frac{dxy}{(xy)^2} + \frac{xy(y \cdot dx - x \cdot dy)}{(xy)^2} = 0$$

(by dividing terms by certain factor make it integrable)

$$\frac{1}{(xy)^2} \cdot d(xy) + \boxed{\frac{y \cdot dx - x \cdot dy}{xy}} = 0 \quad \text{standard.}$$

$$\int \frac{1}{(xy)^2} \cdot d(xy) + \int d\left(\log \frac{x}{y}\right) = 0$$

$$\frac{-1}{xy} + \log \frac{x}{y} = c$$

$$y \cdot dx - x \cdot dy + (1+x^2) \cdot dx + x^2 \sin y \cdot dy = 0$$

$$y \cdot dx - x \cdot dy + (1+x^2) \cdot dx + x^2 \sin y \cdot dy = 0$$

$$\frac{(x \cdot dy - y \cdot dx)}{x^2} + \frac{(1+x^2) \cdot dx}{x^2} + \frac{x^2 \sin y \cdot dy}{x^2} = 0$$

perfect integral factor

$$\boxed{\frac{(x \cdot dy - y \cdot dx)}{x^2}} + \left(\frac{1}{x^2} + 1\right) \cdot dx + \sin y \cdot dy = 0$$

$$-\int d\left(\frac{y}{x}\right) + \int \left(\frac{1}{x^2} + 1\right) \cdot dx + \int \sin y \cdot dy = 0$$

$$\frac{-y}{x} - \frac{1}{x} + x - \cos y = c$$

$$y(axy + e^x) \cdot dx - e^x \cdot dy = 0$$

Perfect integration factor 82

$$\frac{ay \cdot xy \cdot dx}{y^2} + \boxed{\frac{e^x \cdot dx - e^x \cdot dy}{y^2}} = 0$$

$$\int ax \cdot dx + \int d\left(\frac{e^x}{y}\right) = 0$$

$$\frac{ax^2}{2} + \frac{e^x}{y} = C$$

$$(y^2 \cdot e^x + 2xy) \cdot dx - x^2 \cdot dy = 0$$

$$\frac{y^2 \cdot e^x dx}{y^2} + \frac{2xy \cdot dx - x^2 \cdot dy}{y^2} = 0$$

$$\int e^x \cdot dx + \int d\left(\frac{x^2}{y}\right) = 0$$

$$e^x + \frac{x^2}{y} = C$$

$$- \cancel{y} \cdot \cancel{e^x} \cdot 2x^2 y^2 \cdot dx + y \cdot e^x \cdot dx - e^x \cdot dy - y^3 \cdot dy = 0$$

$$\frac{2x^2 y^2 \cdot dx}{y^2} + \frac{y \cdot e^x \cdot dx - e^x \cdot dy}{y^2} - \frac{y^3 \cdot dy}{y^2} = 0$$

$$2 \int x^2 \cdot dx + \int d\left(\frac{e^x}{y}\right) - \int y \cdot dy = 0$$

$$\frac{2x^3}{3} + \frac{e^x}{y} - \frac{y^2}{2} = 0$$

Method 2

If $Mdx + N \cdot dy = 0$ is non-exact but homogenous and if $Mx + Ny \neq 0$ then $\frac{1}{Mx + Ny}$ is Integrating factor.

$$(x^2 + y^2) \cdot dx - 2xy \cdot dy = 0$$

$$\frac{\partial M}{\partial y} = 2y \quad \frac{\partial M}{\partial x} = -2y$$

$$\begin{aligned} Mx + Ny &= (x^2 + y^2) \cdot x + (-2xy) \cdot y \\ &= x^3 + xy^2 - 2xy^2 \neq 0 \end{aligned}$$

then

$$\begin{aligned} \text{Integrating factor, } &= \frac{1}{Mx + Ny} \\ &= \frac{1}{x^3 - xy^2} = \frac{1}{x(x^2 - y^2)} \end{aligned}$$

Any non-exact equation multiplied by its suitable integrating factor, then it becomes exact D.F.

$$\int \frac{x^2 + y^2}{x \cdot (x^2 - y^2)} dx - \frac{2xy}{x(x^2 - y^2)} dy = 0$$

Minor fraction $\frac{d}{dx}$ Major fraction $\frac{d}{dx}$

$$\int \left(\frac{2x}{(x^2 - y^2)} - \frac{1}{x} \right) dx - 0 = \int 0$$

$$\frac{\log(x^2 - y^2)}{x} = \log c$$

$$x^2 - y^2 = cx$$

$$x^2y \cdot dx - (x^3 + y^3) \cdot dy = 0$$

$$\begin{aligned} \text{Integrating factor} &= \frac{1}{Mx + Ny} \\ &= \frac{1}{(x^2y)x + (x^3 + y^3) \cdot y} \\ &= \frac{1}{x^3y - x^3y - y^4} \\ &= \frac{1}{-y^4} \end{aligned}$$

In Integrating factor, importance is not given to sign and constant

$$I.F. = \frac{1}{y^4}$$

$$(x^2y - 2xy^2) dx - (x^3 - 3x^2y) \cdot dy = 0$$

$$\begin{aligned} \text{Integrating factor} &= \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2} \\ &= \frac{1}{x^2y^2} \end{aligned}$$

$$xy \cdot dx - (x^2 + 2y^2) \cdot dy = 0$$

$$\begin{aligned} \text{Integrating factor} &= \frac{1}{x^2y - x^2y - 2y^3} \\ &= \frac{1}{-2y^3} \end{aligned}$$

$$I.F. = \frac{1}{y^3}$$

1.10003.

If $Mdx + Ndy = 0$ is non-exact but is in the form $y \cdot f(x,y) \cdot dx + x \cdot g(x,y) \cdot dy = 0$ and $Mx - Ny \neq 0$, then

$\frac{1}{Mx - Ny}$ is Integrating factor.

$$f(x,y) = x^3 - 5y^2 + 6x + 7y + 8$$

$$g(x,y) = x^3y^3 - 5x^2y^2 + xy + 7$$

$$y(1+xy) \cdot dx + x(1-xy) \cdot dy = 0$$

$$\begin{aligned} Mx - Ny &= xy + x^2y^2 - xy + x^2y^2 \\ &= 2x^2y^2 \neq 0 \end{aligned}$$

$$\therefore \frac{1}{Mx - Ny} = \frac{1}{x^2y^2} \text{ is an integrating factor.}$$

solution is

$$\int \frac{y(1+xy)}{x^2y^2} dx + \int \frac{x(1-xy)}{x^2y^2} dy = 0$$

$$\therefore \int \frac{y}{x^2y^2} + \frac{xy^2}{x^2y^2} dx + \int -\frac{1}{y} dy = 0$$

terms in x removed.

$$\therefore \frac{-1}{xy} + \log\left(\frac{x}{y}\right) = c$$

$$y(x^2y^2 + xy + 1) \cdot dx + x(x^2y^2 - xy + 1) \cdot dy = 0$$

$$\text{Integrating factor} = \frac{1}{x^2y^2}$$

$$y(x^3y^3 + x^2y^2 + xy + 1) \cdot dx + x(x^3y^3 - x^2y^2 - xy + 1) \cdot dy = 0$$

$$y(x^2y^3+2)dx + x(2-x^2y^2)dy = 0$$

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$$\text{Integrating factor} = \frac{1}{x^3y^3}$$

$$y(xy \sin xy + \cos xy)dx + x(2xy \cdot \sin xy - \cos xy)dy = 0$$

$$\text{Integrating factor} = \frac{1}{xy \cos xy}$$

- (i) In both the terms, cancel terms with same degree and same sign.
- (ii) In first bracket, take the remaining terms and increase its degree by 1.
- (iii) Reciprocal of above obtained term is Integrating factor.

Method 4:

If $M \cdot dx + N \cdot dy = 0$ is non-exact i.e. $(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x})$ and if $\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$ is purely function of x then integrating factor is $e^{\int f(x) \cdot dx}$

Method 5:

If $M \cdot dx + N \cdot dy = 0$ is non exact and $\frac{1}{N}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})$ is purely function of y then integrating factor is $e^{\int g(y) \cdot dy}$

Function

Integrating factor

$$1/x$$

$$x$$

$$-1/x$$

$$1/x$$

$$2/x$$

$$x^2$$

$$-2/x$$

$$1/x^2$$

$$3/x$$

$$x^3$$

$$-3/x$$

$$1/x^3$$

$$(x^2 + y^2 + 1) \cdot dx - 2xy \cdot dy = 0$$

$$\frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial x} = -2y$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y + 2y = 4y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -2y - 2y = -4y$$

} single term

M has 3 terms while N has only one term

$$I.F. = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$

$$f(x) = \frac{1}{-2xy} (-4y) = \frac{-2}{x}$$

$$\text{Integrating factor} = 1/x^2$$

$$(y + xy^3) \cdot dx + 2(x^2y^2 + x + y^4) \cdot dy = 0$$

$$\frac{\partial M}{\partial y} = 1 + 3xy^2$$

$$\frac{\partial N}{\partial x} = 2(2xy^2 + 1)$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 + 3xy^2 - 4xy^2 - 2$$

$$= 1 - xy^2$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 4xy^2 + 2 - 1 - 3xy^2$$

$$= xy^2 + 1$$

$$f(x) = \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

$$= \frac{(1 + xy^2)}{y + xy^3}$$

$$= \frac{1}{y} \left(\frac{xy^2 + 1}{xy^2 + 1} \right)$$

$$f(x) = \frac{1}{y}$$

$$I.F. = y$$

$$(y + \frac{y^3}{3} + \frac{x^2}{2}) dx + \frac{1}{4} (x + xy^2) \cdot dy = 0$$

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$$\frac{\partial M}{\partial y} = 1 + y^2$$

$$\frac{\partial N}{\partial x} = \frac{1}{4} + \frac{y^2}{4}$$

∴ difference of $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$ will have two terms

$$f(x) = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$

$$= \frac{1}{\left(\frac{x + xy^2}{4} \right)} \left(1 + y^2 - \frac{1}{4} - \frac{y^2}{4} \right)$$

$$= \frac{4}{x(1+y^2)} \left(\frac{3}{4} y^2 + \frac{3}{4} \right)$$

$$= \frac{3(y^2+1)}{x(1+y^2)}$$

$$= \frac{-3(1+y^2)}{x(1+y^2)} = \frac{-3}{x}$$

$$\therefore I.F. = x^3$$

$$(y^4 + 2y) \cdot dx + (xy^3 + 2y^4 - 4x) dy = 0$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2$$

$$\frac{\partial N}{\partial x} = y^3 - 4$$

$$f(x) = \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

$$= \frac{1}{y^4 + 2y} \left(y^3 - 4 - 4y^3 - 2 \right)$$

$$= \frac{-3y^3 - 6}{y^4 + 2y}$$

$$= \frac{-3(y^3 + 2)}{y(y^3 + 2)}$$

$$= \frac{-3}{y}$$

$$\therefore I.F. = 1/y^3$$

$$(3x^2y^4 + 2xy) dx + (2x^3y^3 - x^2) dy = 0$$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$$

$$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

} difference has + sign
divisible by M
(i.e. having + sign)

$$f(x) = \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

$$= \frac{1}{(3x^2y^4 + 2xy)} (6x^2y^3 - 2x - 12x^2y^3 - 2x)$$

$$= \frac{-6x^2y^3 - 4x}{(3x^2y^4 + 2xy)}$$

$$= \frac{-2(3x^2y^3 + 2x)}{y(3x^2y^3 + 2x)}$$

$$= \frac{-2}{y}$$

$$\therefore \text{I.P.} = \frac{1}{y^2}$$

Linear equation:

A differential equation is of the form $\frac{dy}{dx} + Py = Q$ where P and Q are function of x , is said to be function of x .

Then integrating factor = $e^{\int P \cdot dx}$

General solution is

$$y \cdot e^{\int P \cdot dx} = \int (Q \cdot e^{\int P \cdot dx}) \cdot dx + c$$

For $\frac{dx}{dy} + Px = Q$.

$$\text{G.S. } x \cdot e^{\int P \cdot dy} = \int (Q \cdot e^{\int P \cdot dy}) \cdot dy + c$$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{\log x}{x}$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right) \cdot y = \frac{\log x}{x}$$

$$y(x) = \int \frac{\log x}{x} \cdot (x) \cdot dx + c$$

$$xy = x \cdot \log x - x + c$$

$$(1-x^2) \cdot \frac{dy}{dx} - 2xy = x^2$$

$$\frac{dy}{dx} - x^2 \cdot \frac{dy}{dx} - 2xy = x^2$$

$$\frac{dy}{dx} - \left(\frac{2x}{1-x^2}\right) \cdot y = \frac{x^2}{1-x^2}$$

$$\begin{aligned} \text{I.F.} &= e^{\int P \cdot dx} = e^{\int \frac{-2x}{1-x^2} \cdot dx} \\ &= e^{\log(1-x^2)} \\ &= 1-x^2 \end{aligned}$$

General solution is

$$\begin{aligned} y(1-x^2) &= \int \frac{x^2}{(1-x^2)} \cdot (1-x^2) \cdot dx + c \\ &= \frac{x^3}{3} + c \end{aligned}$$

$$x \log x \frac{dy}{dx} + 2xy = e^{-xy} \quad y(0) = 1$$

$$\frac{dy}{dx} + (2x) \cdot y = e^{-xy}$$

$$\begin{aligned} \text{I.F.} &= e^{\int 2x \cdot dx} \\ &= e^{x^2} \end{aligned}$$

solution:

$$y \cdot (e^{x^2}) = \int e^{-x^2} \cdot e^{x^2} \cdot dx + c$$

$$y \cdot e^{x^2} = x + c$$

$$1 \cdot e^0 = 0 + c$$

$$y(0) = 1$$

$$c = 1$$

$$y \cdot e^{x^2} = x + 1$$

$$y = \frac{x+1}{e^{x^2}}$$

$$y = e^{-x^2} \cdot (x+1)$$

$$x \cdot \log x \cdot \frac{dy}{dx} + y = 2 \log x$$

$$\frac{dy}{dx} + \left(\frac{1}{x \cdot \log x} \right) \cdot y = \frac{2 \log x}{x}$$

$$I.F. = e^{\int \frac{1}{x \log x} \cdot dx}$$

$$= e^{\int \frac{\frac{1}{x}}{\log x} \cdot dx}$$

$$= e^{\log(\log x)}$$

$$= \log x$$

solution:

$$y (\log x) = \int \frac{2 \log x}{x} \cdot dx + c$$

$$= 2 \int \log x \cdot \frac{1}{x} \cdot dx + c$$

$$= 2 \frac{(\log x)^2}{2} + c$$

$$y = \log x + \frac{c}{\log x}$$

$$(x+2y^3) \cdot \frac{dy}{dx} = y$$

(88)

$$\frac{dx}{dy} = \left(\frac{x+2y^3}{y} \right)$$

∴ linear in x

$$\frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\frac{dx}{dy} - \left(\frac{1}{y} \right) \cdot x = 2y^2$$

$$I.F. = \frac{1}{y}$$

General studies,

$$x \left(\frac{1}{y} \right) = \int 2y^2 \cdot \frac{1}{y} \cdot dy + c$$

$$\frac{x}{y} = y^2 + c$$

$$x = y^3 + cy$$

$$(1+y^2) \cdot dx + (x - e^{\tan^{-1}y}) \cdot dy = 0$$

$$\frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial x} = 1 - 0$$

Method I -

Method II - Not homogenous

Method III - No. xy term

Method IV & V } - Not divisible by M & N also

$$\frac{dx}{dy} = \frac{e^{\tan^{-1}y}}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{1}{1+y^2} \cdot x + \frac{dx}{dy} = \frac{e^{\tan^{-1}y}}{1+y^2}$$

which is linear.

$$I.F. = e^{\int \frac{1}{1+y^2} \cdot dy}$$

$$= e^{\tan^{-1}y}$$

solution,

$$\begin{aligned} x \cdot e^{\tan^{-1} y} &= \int \frac{e^{\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1} y} dy + c \\ &= \frac{1}{2} \int e^{2 \tan^{-1} y} \cdot \frac{2}{1+y^2} dy + c \\ x \cdot e^{\tan^{-1} y} &= \frac{1}{2} \cdot e^{2 \tan^{-1} y} + c. \end{aligned}$$

Bernoulli's equation:

A D.E. of the form $\frac{dy}{dx} + P \cdot y = Q \cdot y^n$, where P & Q are functions of x , is said to be Bernoulli's equation.

$$\frac{dy}{dx} + P \cdot y = Q \cdot y^n$$

$$\frac{1}{y^n} \frac{dy}{dx} + \frac{P \cdot y}{y^n} = \frac{Q \cdot y^n}{y^n}$$

$$y^{-n} \frac{dy}{dx} + P \cdot y^{1-n} = Q$$

$$Z = y^{1-n}$$

$$\frac{dz}{dx} = (1-n) \cdot y^{1-n-1} \cdot \frac{dy}{dx}$$

$$y^{-n} \frac{dy}{dx} = \frac{1}{(1-n)} \cdot \frac{dz}{dx}$$

$$P \cdot Z - Q = \frac{1}{(1-n)} \cdot \frac{dz}{dx}$$

$$\frac{1}{(1-n)} \frac{dz}{dx} + PZ = Q$$

$$\frac{dz}{dx} + P(1-n) \cdot Z = Q \cdot (1-n) \text{ which is linear.}$$

$$I.F. = e^{\int P(1-n) \cdot dx}$$

General solution is

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$$y^{1-n} \cdot e^{\int p(1-n) \cdot dx} = \int (Q \cdot (1-n) \cdot e^{\int p(1-n) \cdot dx}) \cdot dx + C$$

JF.

$$\frac{dx}{dy} + px = Q \cdot x^n$$

$$\text{Integrating factor} = e^{\int p(1-n) \cdot dy}$$

General solution.

$$x^{1-n} \cdot e^{\int p(1-n) \cdot dy} = \int [Q \cdot (1-n) \cdot e^{\int p(1-n) \cdot dy}] \cdot dy + C$$

$$\frac{dy}{dx} + \frac{y}{x} = x^2 \cdot y^3$$

$$\begin{aligned} \text{J.F.} &= e^{\int p(1-n) \cdot dx} \\ &= e^{\int \frac{1}{x} (1-3) \cdot dx} \\ &= e^{-\int \frac{2}{x} \cdot dx} \\ &= \frac{1}{x^2} \end{aligned}$$

$$\frac{dy}{dx} \cdot (xy + x^2 y^3) = 1$$

$$\frac{dy}{dx} = xy + x^2 \cdot y^3$$

$$\frac{dx}{dy} - yx = y^3 \cdot x^2$$

$$\begin{aligned} \text{J.F.} &= e^{\int -y(1-2) \cdot dy} \\ &= e^{\int y \cdot dy} \\ &= e^{y^2/2} \end{aligned}$$

Orthogonal trajectory :

The family of curves which cuts every member of given family orthogonally is called Orthogonal trajectory of given family.

$$f(x, y, c) = 0$$

— (i)

To remove c , differentiate

$$F(x, y, \frac{dy}{dx}) = 0$$

— (ii)

To get orthogonal trajectory, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$g(x, y, -\frac{dx}{dy}) = 0$$

— (iii)

The solution of equⁿ — (iii) is called orthogonal trajectory of given family.

(i) If equation — (i) & — (iii) are identical then the curve is said to be self orthogonal. (perpendicular to itself)

Q. Find the orthogonal trajectory of family of lines passing through origins. is.

- (a) family of lines
- (b) family of circles having centre at origin
- (c) family of parabolas
- (d) family of ellipses.

equⁿ of lines passing through origin

$$y = mx + c$$

— (i)

$$\frac{dy}{dx} = m$$

$$y = x \cdot \frac{dy}{dx}$$

— (ii)

Replace $\frac{dy}{dx}$ by $(-\frac{dx}{dy})$

(40)

$$y = -x \cdot \frac{dx}{dy}$$

— (ii)

$$\int y \cdot dy = \int -x \cdot dx$$

$$\frac{y^2}{2} + \frac{x^2}{2} = c$$

$$x^2 + y^2 = 2c$$

i.e.

$$x^2 + y^2 = r^2$$

— equation of circle with
centre origin

Find trajectory of family of parabolas $y^2 = 4ax$

$$y^2 = 4ax$$

— (i)

$$2y \cdot \frac{dy}{dx} = 4a$$

$$y^2 = 2y \cdot \frac{dy}{dx} \cdot x$$

$$y = 2x \cdot \frac{dy}{dx}$$

— (ii)

replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$y = 2x \left(-\frac{dx}{dy} \right)$$

$$\int y \cdot dy = \int -2x \cdot dx$$

$$\frac{y^2}{2} + x^2 = c$$

$$\frac{x^2}{c} + \frac{y^2}{2c} = -1$$

— equation of ellipse

Find trajectory of hyperbola $x^2 - y^2 = a^2$ - rectangular hyperbola

$$x^2 - y^2 = a^2$$

— (i)

$$2x - 2y \cdot \frac{dy}{dx} = 0$$

$$x - y \cdot \frac{dy}{dx} = 0$$

— (ii)

replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$x + y \cdot \frac{dx}{dy} = 0$$

$$x \cdot dy + y \cdot dx = 0$$

$$\int d(xy) = \int 0$$

$$xy = c$$

equⁿ of asymptote

Find trajectory for parabola $y^2 = 4a(x+a)$

$$y^2 = 4a(x+a)$$

— (i)

$$2y \cdot \frac{dy}{dx} = 4a$$

$$a = \frac{y \cdot y_1}{2}$$

$$\therefore y^2 = \frac{4y \cdot y_1}{2} \cdot \left(x + \frac{y \cdot y_1}{2}\right)$$

$$y = 2x y_1 + y \cdot y_1^2$$

— (ii)

replace y_1 by $-\frac{1}{y_1}$

$$y = 2x \cdot \frac{-1}{y_1} + y \cdot \frac{1}{y_1^2}$$

$$y = \frac{-2xy_1 + y}{y_1^2}$$

$$y \cdot y_1^2 = -2xy_1 + y$$

$$y = 2xy_1 + y y_1^2$$

— (iii)

∴ — (d) & — (ii) are same

(9)

Given curve is self orthogonal.

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \quad \text{self orthogonal,}$$

* first order D.E. has only one arbitrary constant.

Orthogonal trajectory of $x^2 + y^2 + 2gx - 5 = 0$ is

$$x^2 + y^2 - 2Py + 5 = 0$$

Monday
30th September 201

Differential equation of 1st order but not 1st degree :

A D.E. is of the form :

$$a_0 \left(\frac{dy}{dx} \right)^n + a_1 \left(\frac{dy}{dx} \right)^{n-1} + a_2 \left(\frac{dy}{dx} \right)^{n-2} + \dots + a_n = 0$$

where $a_0, a_1, a_2, \dots, a_n$ are functions of x & y and constants, is said to be D.E. of 1st order but not first degree.

$$\text{Let } \frac{dy}{dx} = p$$

$$\therefore a_0 \cdot p^n + a_1 p^{n-1} + a_2 p^{n-2} + \dots + a_n = 0$$

$$\text{Let } f(x, y, p) = 0$$

Thus, equation containing three variables x, y & p

It may contain three types of solutions

- solvable for p
- solvable for x
- solvable for y

If solvable for p , then it must be factorised,

$$[p - f_1(x, y)] [p - f_2(x, y)] [p - f_3(x, y)] \dots [p - f_n(x, y)]$$

$$p - f_1(x, y) = 0$$

$$\frac{dy}{dx} = f_1(x, y)$$

$$p^2 - 3p + 6 = 0 \quad \text{where } p = \frac{dy}{dx}$$

$$(p-2)(p+3) = 0$$

$$p-2 = 0$$

$$\frac{dy}{dx} - 2 = 0$$

$$\int dy = 2 \int dx$$

$$y - 2x - C_1 = 0$$

$$\therefore \text{soln is } (y - 2x - C_1)(y - 3x - C_2) = 0$$

$$1. x^2 p^2 + 2xy p - 6y^2 = 0$$

$$(x^2 p^2 + 3xy p - 2xy p - 6y^2) = 0$$

$$xp(xp + 3y) - 2y(xp + 3y) = 0$$

$$(xp + 3y)(xp - 2y) = 0$$

If $x \cdot \frac{dx}{dy}$ or $y \cdot \frac{dy}{dx}$ is there go for variable separable.

If $x \cdot \frac{dy}{dx} \therefore y \cdot \frac{dx}{dy}$ go for linear.

$$xp + 3y = 0$$

$$x \cdot \frac{dy}{dx} + 3y = 0$$

$$\frac{dy}{dx} + \left(\frac{3}{x}\right)y = 0$$

$$y \cdot e^{\int \frac{3}{x} dx} (x^3 y - C_1)(y - C_2 x^2) = 0$$

$$x \cdot \frac{dy}{dx} - 2y = 0$$

$$\frac{dy}{dx} - \frac{2}{x} y = 0$$

$$y \left(\frac{1}{x^2}\right) - C_2 = 0$$

$$2. xy p^2 + xy p - 6y^2 = 0$$

$$xy p^2 - x^2 p - y^2 p + xy = 0$$

$$xp(y p - x) - y(y p - x) = 0$$

$$(yp - x)(xp - y) = 0$$

$$(y p - x) = 0$$

$$y \cdot \frac{dy}{dx} - x = 0$$

$y \frac{dy}{dx}$ - go for variable separable

$$\int y \cdot dy - \int x \cdot dx = \int 0$$

$$\frac{y^2}{2} - \frac{x^2}{2} = c_1$$

$$y^2 - x^2 - 2c_1 = 0$$

$$(x p - y) = 0$$

$$x \cdot \frac{dy}{dx} - y = 0$$

$$\frac{dy}{dx} - \left(\frac{1}{x}\right) \cdot y = 0$$

$$y \left(\frac{1}{x}\right) - c_2 = 0$$

$$(y^2 - x^2 - 2c_1) \left(-\frac{y}{x} - c_2\right) = 0$$

solvable for y

solvable for x

(i) degree of y must be 1

(i) degree for x must be 1

(ii) Separate y from x and p

(ii) Separate x from y and p

$$y = f(x, p)$$

$$x = f(y, p)$$

(iii) Differentiating on both sides w.r.t. x

(iii) Differentiating on both sides w.r.t. y

(iv) Replace $\frac{dy}{dx}$ by p.

(iv) Replace $\frac{dx}{dy}$ by $\left(\frac{1}{p}\right)$

(v) Neglect the terms or cancel the terms which does not contain $\frac{dp}{dx}$

(v) Neglect the terms which does not contain $\frac{dp}{dy}$

$$1. y^2 + 2px = p^2 x^4$$

$$y + px = p^2 x^4$$

$$\text{where } p = \frac{dy}{dx}$$

(there are only three terms, all having coefficient 1.
thus factorisation not possible - Not solvable for P)

$$y = p^2 x^4 - px$$

$$\frac{dy}{dx} = p^2 (4x^3) + x^4 (2p \cdot \frac{dp}{dx}) - p(1) - x \cdot \frac{dp}{dx}$$

$$p - 4p^2 x^3 + p = x^4 (2p \cdot \frac{dp}{dx}) - x \cdot \frac{dp}{dx}$$

$$2p(1 - 2px^3) = -x(1 - 2px^3) \cdot \frac{dp}{dx}$$

$$2p = -x \cdot \frac{dp}{dx}$$

by variable - separable

$$\int \frac{-2 dx}{x} = \int \frac{dp}{p}$$

$$\text{Borrow: } \log(C/x^2) = \therefore \log p$$

$$p = \frac{C}{x^2}$$

— (solⁿ should be
y in terms of x)

$$y + x \cdot \left(\frac{C}{x^2}\right) = \left(\frac{C}{x^2}\right)^2 x^4$$

$$y + \frac{C}{x} = C^2$$

— (solution)

$$\frac{dy}{dx} = \frac{C}{x^2}$$

$$\int dy = \int \frac{C}{x^2} \cdot dx$$

$$y = -\frac{C}{x} + C'$$

$$y + \frac{C}{x} = C'$$

— (solution)

$$2. \quad y = 2px + y^2 p^3$$

$$2px = y - y^2 p^3$$

$$2x = \frac{y}{p} - y^2 p^2$$

$$2x = \left(\frac{1}{p}\right) \cdot y - y^2 p^2$$

$$2 \cdot \frac{dx}{dy} = \left(\frac{1}{p}\right) + y \left(\frac{-1}{p^2}\right) \frac{dp}{dy}$$

If eqn is in the form $y = px + \phi(p)$

$$2P - \frac{1}{P} - p^2 \cdot 2y = -\frac{1}{p^2} \cdot y \cdot \frac{dp}{dy} - y^2 \left(2p \frac{dp}{dy}\right)$$

Replace p by c
solution is

$$-P \left(2 + \frac{1}{p^2} + 2py\right) = -yP \frac{dp}{dy} \left(\frac{1}{p^2} + 2y \cdot p\right)$$

$y = cx + \phi(c)$
C general s

$$P = yP \cdot \frac{dp}{dy}$$

$$\int \frac{dy}{y} = \int dp$$

$$\log y = p + c$$

Differential equation with constant coefficients:

A D.E. is of the form,

$$a_0 \cdot \frac{d^2 y}{dx^2} + a_1 \cdot \frac{dy^{n-1}}{dx^{n-1}} + a_2 \cdot \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = Q$$

where $a_0, a_1, \dots, a_n$ are constants and Q be function of x , is said to be D.E. with constant coefficients.

$$\frac{d}{dx} = D$$

$$a_0 \cdot D^2 y + a_1 D^{n-1} y^{n-1} + a_2 \cdot D^{n-2} y^{n-2} + \dots + a_n y = Q$$

$$f(D)y = Q$$

This equⁿ may contain two types of solⁿ. One is called complimentary function other is called particular integral.

Note:

(i) If $Q=0$, then complimentary function is called the general solution.

(ii) If $Q \neq 0$, then only particular integral existed.

$$G.S. = C.F. + P.I.$$

1. To find the complimentary function.

(i) Replace D by m , $f(m)=0$ is called auxillary equⁿ.

$$a_0 \cdot m^2 + a_1 m^{n-1} + a_2 \cdot m^{n-2} + \dots + a_n = 0$$

Case I: The roots are real and unequal.

$$y_c = c_1 \cdot e^{\alpha x} + c_2 \cdot e^{\beta x} + c_3 \cdot e^{\gamma x} + \dots$$

where α, β, γ are roots of equⁿ.

e.g.

$$\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0$$

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$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

$$\therefore y = C_1 \cdot e^{2x} + C_2 \cdot e^{3x}$$

Case II: The roots are real and equal.

$$m = \alpha, \alpha, \beta$$

$$y_c = (C_1 + C_2 x) \cdot e^{\alpha x} + C_3 \cdot e^{\beta x}$$

$$\text{If } m = \alpha, \alpha, \alpha, \beta$$

$$y_c = (C_1 + C_2 x + C_3 x^2) \cdot e^{\alpha x} + C_4 \cdot e^{\beta x}$$

$$\text{If } m = \alpha, \alpha, \beta, \beta$$

$$y_c = (C_1 + C_2 x) e^{\alpha x} + (C_3 + C_4 x) e^{\beta x}$$

e.g.

$$\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

solⁿ

$$y = (C_1 + C_2 x) \cdot e^{2x}$$

Case III: If the roots are imaginary.

$$\text{If } m = \alpha \pm i\beta$$

$$y = [C_1 \cos \beta x + C_2 \sin \beta x] e^{\alpha x}$$

$$\text{If } m = \alpha \pm i\beta, \alpha \pm i\beta$$

$$y = [(C_1 + C_2 x) \cos \beta x + (C_3 + C_4 x) \sin \beta x] e^{\alpha x}$$

e.g.

$$\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$$

$$m^2 - 2m + 2 = 0$$

$$m = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= (1 \pm i)$$

$$y = (C_1 \cos x + C_2 \sin x) \cdot e^x$$

2. To find the roots of the equation.

- (i) Every third degree eqn must contain at least one real root.
- (ii) If sum of all the coefficients is zero then $(m=1)$ satisfies the equation or $(m-1)$ is a factor.
- (iii) If sum of coefficients of even power equal to sum of coefficients of odd power then $m=-1$ satisfies the equation or $(m+1)$ is factor.

e.g.

$$(D^3 - 3D + 2)y = 0$$

$$m^3 - 3m + 2 = 0$$

$$\text{sum of coefficients} = 1 - 3 + 2 = 0$$

$m=1$ is root

By Hobner's synthetic division

	m^3	m^2	m	con
	1	0	-3	2
$m=1$	0	1	1	-2
	1	1	-2	0

Q5

	1	1	-2	0
m=1	0	1	2	
	1	2	0	

$$m+2=0$$

$$m=-2$$

solⁿ is

$$y = (C_1 + C_2 x) \cdot e^x + C_3 \cdot e^{-2x}$$

1. $(D^3 - 4D^2 + 5D - 2)y = 0$

$$m^3 - 4m^2 + 5m - 2 = 0$$

$$\text{sum of coeff.} = 1 - 4 + 5 - 2 = 0$$

$m=1$ is root of eqn

	1	-4	5	-2
m=1	0	1	-3	2
	1	-3	2	0
m=1	0	1	-2	
	1	-2	0	

$$m-2=0$$

$$m=2$$

solⁿ is

$$y = (C_1 + C_2 x) \cdot e^x + C_3 \cdot e^{2x}$$

2. $(D^4 - 2D^3 + 2D - 1)y = 0$

$$m^4 - 2m^3 + 2m - 1 = 0$$

$m=1$ is root of eqn

	1	-2	0	2	-1
m=1	0	1	-1	-1	1
	1	-1	-1	1	0
m=1	0	1	0	-1	
	1	0	-1	0	
m=1	0	1	1		
	1	1	0		

$$m-1=0$$

$$m=1$$

Solⁿ is

$$y = C_1 + C_2 x + C_3 x \cdot e^x + C_4 x \cdot e^{-x}$$

$$3 \cdot (D^3 + 6D^2 + 11D + 6)y = 0$$

$$m^3 + 6m^2 + 11m + 6 = 0$$

$$\text{sum of coeff. of odd powers} = 1 + 11 = 12$$

$$\text{even powers} = 6 + 6 = 12$$

$m = -1$ root of equation.

	1	6	11	6
$m = -1$	0	-1	-5	-6
	1	5	6	0

$$m^2 + 5m + 6 = 0$$

$$(m+2)(m+3) = 0$$

$$m = -2, -3$$

$$y = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{-3x}$$

$$4. (D^4 - D^3 - 9D^2 - 11D - 4)y = 0$$

$$(m^4 - m^3 - 9m^2 - 11m - 4) = 0$$

$$\text{sum of coeff. of odd powers} = -1 - 11 = -12$$

$$\text{even powers} = 1 - 9 - 4 = -12$$

$m = -1$ root of eqn

	1	-1	-9	-11	-4
$m = -1$	0	-1	2	-7	-4
	1	-2	-7	-4	0
$m = -1$	0	-1	3	4	
	1	-3	-4	0	
$m = -1$	0	-1	-4		
	1	-2	0		

$$\text{odd} = 1 - 7 = -6$$

$$\text{even} = -2 - 4 = -6$$

$$\text{odd} = 1 - 4 = -3$$

$$\text{even} = -3$$

$$m-4=0$$

$$m=4.$$

(8)

$$y = (c_1 + c_2 x + c_3 x^2) e^{-x} + c_4 e^{4x}.$$

Q.

$$m^4 + 2m^2 + 1 = 0$$

$$(m^2 + 1)^2 = 0$$

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm i, \pm i$$

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$$y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x.$$

3. To find particular integral :

$$f(D)y = Q.$$

$$y_p = P.I. = \frac{1}{f(D)} Q$$

(~~XXXX~~ A.T.E.)

Q may be A.T

$$\frac{1}{D-\alpha} Q = e^{\alpha x} \int Q \cdot e^{-\alpha x} dx.$$

e.g.

$$\frac{1}{D^2 - 3D + 2} e^{3x}$$

$$= \frac{1}{(D-2)(D-1)} e^{3x}$$

$$= \frac{1}{D-1} \cdot e^{2x} \int e^{3x} \cdot e^{-2x} dx$$

$$= \frac{1}{D-1} \cdot e^{2x} \int e^x dx$$

$$= \frac{1}{D-1} \cdot e^{2x} \cdot e^x$$

$$\begin{aligned}
 &= e^x \int e^{3x} \cdot e^{-x} \cdot dx \\
 &= e^x \int e^{2x} \cdot dx \\
 &= \frac{e^x \cdot e^{2x}}{2} \\
 &= \frac{e^{3x}}{2}
 \end{aligned}$$

is solution of equation
(particular integral)

Case I: To find P.I. of form $\frac{1}{f(D)} \cdot e^x$

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} \cdot e^{ax} \quad \text{if } f(a) \neq 0$$

e.g.

$$\begin{aligned}
 \frac{1}{D^2 - 3D + 2} e^{3x} &= \frac{1}{9 - 3(3) + 2} e^{3x} \\
 &= \frac{e^{3x}}{2}
 \end{aligned}$$

Case II:

$$\begin{aligned}
 \text{If } \frac{1}{f(D)} e^{ax} &= \frac{x}{f'(D)} \cdot e^{ax} \quad \text{if } f(a) = 0 \\
 &= \frac{x^2}{f''(D)} \cdot e^{ax} \quad \text{if } f'(a) = 0 \\
 &= \frac{x^3}{f'''(D)} \cdot e^{ax} \quad \text{if } f''(a) = 0
 \end{aligned}$$

$$1. (D^2 - 7D + 6) \cdot y = e^{6x}$$

$$y = \frac{1}{D^2 - 7D + 6} \cdot e^{6x}$$

$$= \frac{x}{2D - 7} \cdot e^{6x}$$

$$= \frac{x}{2(6) - 7} \cdot e^{6x}$$

$$= \frac{x}{5} \cdot e^{6x}$$

$$2. (D^2 + 4D + 4)y = e^{2x} + e^{-2x}$$

$$y = \frac{1}{D^2 + 4D + 4} \cdot e^{2x} + e^{-2x}$$

$$= \frac{1}{D^2 + 4D + 4} \cdot e^{2x} + \frac{1}{D^2 + 4D + 4} \cdot e^{-2x}$$

$$= \frac{1}{2^2 + 4(2) + 4} \cdot e^{2x} + \frac{1}{4 + 4(-2) + 4} \cdot e^{-2x}$$

$$= \frac{1}{16} \cdot e^{2x} + \frac{x}{2D + 4} \cdot e^{-2x}$$

$$= \frac{1}{16} \cdot e^{2x} + \frac{x^2}{2(1)} \cdot e^{-2x}$$

$$= \frac{1}{16} \cdot e^{2x} + \frac{x^2 \cdot e^{-2x}}{2}$$

$$3. (D^2 - 3D + 2)y = \cosh x$$

$$= \frac{1}{2} \left[\frac{1}{D^2 - 3D + 2} \cdot e^x + \frac{1}{D^2 - 3D + 2} \cdot e^{-x} \right]$$

$$= \frac{1}{2} \left[\frac{x}{2D - 3} \cdot e^x + \frac{1}{1 + 3 + 2} \cdot e^{-x} \right]$$

$$= \frac{1}{2} \left[-x \cdot e^x + \frac{1}{6} \cdot e^{-x} \right]$$

$$= -\frac{x}{2} e^x + \frac{1}{12} e^{-x}$$

$$4. (D^4 - D^3 - 9D^2 - 11D - 4). y = e^{-x}$$

$$= \frac{1}{D^4 - D^3 - 9D^2 - 11D - 4} e^{-x}$$

$$= \frac{x}{4D^3 - 3D^2 - 18D - 11} e^{-x}$$

$$= \frac{x^2}{12D^2 - 6D - 18} e^{-x}$$

$$= \frac{x^3}{24D - 6} e^{-x}$$

$$= \frac{x^3}{24(-1) - 6} e^{-x}$$

$$= \frac{-x^3}{30} e^{-x}$$

Case III: To find P.I. of form $\frac{1}{f(D)} \sin ax$, $\frac{1}{f(D)} \cos ax$,
replace D^2 by $-a^2$ if $\phi(-a^2) \neq 0$.

e.g. $\frac{1}{D^2 - 2D + 5} \sin 2x$

$$= \frac{1}{(-a)^2 + 2D + 5} \sin 2x$$

$$= \frac{1}{-4 + 2D + 5} \sin 2x$$

$$= \frac{1}{1 - 2D} \sin 2x$$

$$= \frac{(1 + 2D)}{(1 - 2D)(1 + 2D)} \sin 2x$$

$$= \frac{(1 + 2D)}{1 - 4D^2} \sin 2x$$

$$= \frac{(1 + 2D)}{1 - 4(-4)} \sin 2x$$

$$\begin{aligned}
 &= \frac{1+2D}{17} \cdot \sin 2x \\
 &= \frac{\sin 2x \cdot 2D(\sin 2x)}{17} \\
 &= \frac{\sin 2x \cdot 2 \times 2 \cdot \cos 2x}{17} \\
 &= \frac{\sin 2x \cdot 4 \cdot \cos 2x}{17}
 \end{aligned}$$

Case IV: To find p.i. of form $\frac{1}{D^2+a^2} \cdot \sin ax$, $\frac{1}{D^2+a^2} \cdot \cos ax$

$$\begin{aligned}
 \frac{1}{D^2+a^2} \cdot \sin ax &= \frac{x}{2D} \cdot \sin ax \\
 &= \frac{x}{2} \left[\frac{-\cos ax}{a} \right] \quad \begin{array}{l} D = \text{derivative} \\ \frac{1}{D} = \text{integration} \end{array} \\
 &= \frac{-x}{2a} \cdot \cos ax
 \end{aligned}$$

$$\frac{1}{D^2+a^2} \cdot \cos ax = \frac{x}{2a} \cdot \sin ax$$

e.g. $\frac{1}{D^2+4} \cdot \cos 2x = \frac{x}{4} \cdot \sin 2x$

$$\frac{1}{D^2+9} \cdot \cos 3x = \frac{x}{6} \cdot \sin 3x$$

$$\frac{1}{D^2+16} \cdot \sin 4x = \frac{-x}{8} \cdot \cos 4x$$

* $\frac{1}{D^2+1} \cdot \sin 2x = \frac{1}{-4+1} \sin 2x = \frac{-1}{3} \sin 2x$

$$1. (D^2 - D - 2)y = \cos 2x$$

$$P.J. = \frac{1}{D^2 - D - 2} \cos 2x$$

$$= \frac{1}{-4 - D - 2} \cos 2x$$

$$= \frac{1}{-6 - D} \cos 2x$$

$$= \frac{-6 + D}{(-6 - D)(-6 + D)} \cos 2x$$

$$= \frac{-6 + D}{36 - D^2} \cos 2x$$

$$= \frac{-6 + D}{36 + 4} \cos 2x$$

$$= \frac{-6 \cos 2x - 2 \sin 2x}{40}$$

$$2. (D^2 - 4)y = \sin^2 x$$

$$P.J. = \frac{1}{D^2 - 4} \sin^2 x$$

$$= \frac{1}{D^2 - 4} \left(\frac{1 - \cos 2x}{2} \right)$$

$$= \frac{1}{2} \left[\frac{1}{D^2 - 4} \underset{= e^{0x}}{(1)} - \frac{1}{D^2 - 4} \cos 2x \right]$$

$$= \frac{1}{2} \left[\frac{1}{0 - 4} - \frac{1}{-4 - 4} \cos 2x \right]$$

$$= \frac{1}{2} \left[-\frac{1}{4} + \frac{1}{8} \cos 2x \right]$$

$$= -\frac{1}{8} + \frac{1}{16} \cos 2x$$

$$3. (D^2+1)y = \sin 2x \cdot \cos x.$$

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$$P.I. = \frac{1}{D^2+1} \sin 2x \cdot \cos x$$

$$= \frac{1}{2} \cdot \frac{1}{D^2+1} 2 \sin 2x \cdot \cos x$$

$$= \frac{1}{2} \left[\frac{1}{D^2+1} \sin 3x + \frac{1}{D^2+1} \sin x \right] \quad \begin{matrix} 2 \sin A \cos B \\ = \sin(A+B) \\ \sin(A-B) \end{matrix}$$

$$= \frac{1}{2} \left[\frac{1}{-9+1} \sin 3x + \frac{x}{2(1)} \cos x \right]$$

$$= \frac{-1}{16} \sin 3x - \frac{x}{4} \cos x.$$

$$4. (D^2-2D+5)y = \cos 3x.$$

$$P.I. = \frac{1}{D^2-2D+5} \cos 3x$$

$$= \frac{1}{-9-2D+5} \cos 3x$$

$$= \frac{1}{-4-2D} \cos 3x$$

$$= \frac{-4+2D}{16-4D^2} \cos 3x$$

$$= \frac{-4+2D}{16+36} \cos 3x$$

$$= \frac{1}{52} (-4 \cos 3x + 6 \sin 3x)$$

Case V: To find P.I. of the form $\frac{1}{f(D)} x^m$.

$$\frac{1}{f(D)} x^m = [f(D)]^{-1} \cdot x^m$$

$$(1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$$

$$(1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$$

$$(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \dots$$

$$(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \dots$$

1. $(D^2 - 4)y = x^2$

$$\begin{aligned} P.I. &= \frac{1}{D^2 - 4} \cdot x^2 \\ &= \frac{1}{-4 \left(1 - \frac{D^2}{4}\right)} \cdot x^2 \end{aligned}$$

$$= -\frac{1}{4} \left[1 - \frac{D^2}{4}\right]^{-1} \cdot x^2$$

$$= -\frac{1}{4} \left[1 + \frac{D^2}{4} + \frac{D^4}{16} + \dots\right] \cdot x^2$$

$$= -\frac{1}{4} \left(x^2 + \frac{2}{4}\right)$$

$$= -\frac{1}{4} \left(x^2 + \frac{1}{2}\right)$$

In $f(D)$, if algebraic function, any const. other than 1 is taken outside.

2. $(D^3 + 8)y = x^4 + 2x + 1$

$$P.I. = \frac{1}{D^3 + 8} (x^4 + 2x + 1)$$

$$= \frac{1}{8 \left(1 + \frac{D^3}{8}\right)} (x^4 + 2x + 1)$$

$$= \frac{1}{8} \left(1 + \frac{D^3}{8}\right)^{-1} (x^4 + 2x + 1)$$

$$= \frac{1}{8} \left[1 - \frac{D^3}{8} \right] \cdot (x^4 + 2x + 1)$$

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$$= \frac{1}{8} \left[x^4 + 2x + 1 - \frac{1}{8} (24x + 0 + 0) \right]$$

$$= \frac{1}{8} (x^4 + 2x + 1 - 3)$$

$$= \frac{x^4 + 2x - 2}{8}$$

3. $(D^2 + 2D + 1)y = x^2 + 2x$

$$P.I. = \frac{1}{(1+D)^2} \cdot x^2 + 2x$$

$$= (1+D)^{-2} \cdot (x^2 + 2x)$$

$$= (1 - 2D + 3D^2) \cdot (x^2 + 2x)$$

$$= [x^2 + 2x - 2(2x + 2) + 3(2 + 0)]$$

$$= [x^2 + 2x - 4x - 4 + 6]$$

$$= (x^2 - 2x + 2)$$

Case VI To find P.I. of form $\frac{1}{f(D)} \cdot e^{ax} \cdot v$ where v is a function of x .

$$\frac{1}{f(D)} e^{ax} v = e^{ax} \frac{1}{f(D+a)} \cdot v$$

v -can be trigonometric or algebraic but NOT exponential

e.g.

1. $(D^2 - 2D + 1)y = x^2 \cdot e^x$

$$P.I. = e^x \cdot \frac{1}{(D-1)^2} \cdot x^2$$

$$= e^x \frac{1}{(D+1-1)^2} \cdot x^2$$

$$= e^x \frac{1}{D^2} \cdot x^2$$

$$= e^x \cdot \frac{x^2}{12}$$

$$\frac{1}{D^2} = \int \int$$

$$2. (D^2 - 2D + 5)y = e^x \sin x$$

$$P.J. = \frac{1}{(D^2 - 2D + 5)} \cdot e^x \sin x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 5} \sin x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 5} \sin x$$

$$= e^x \frac{1}{D^2 + 4} \sin x$$

$$= e^x \frac{1}{-1 + 4} \sin x$$

$$= e^x \frac{\sin x}{3}$$

$$3. (D^2 - 4D + 4)y = e^{2x} \cos 3x$$

$$P.J. = \frac{1}{D^2 - 4D + 4} \cdot e^{2x} \cos 3x$$

$$= e^{2x} \frac{1}{(D-2)^2} \cos 3x$$

$$= e^{2x} \frac{1}{(D+2-2)^2} \cos 3x$$

$$= e^{2x} \frac{1}{D^2} \cos 3x$$

$$= e^{2x} \frac{-\cos 3x}{9}$$

Case VII: To find P.I. of the form $\frac{1}{f(D)} \cdot x^v$ where v is function of x . (101)

$$\frac{1}{f(D)} \cdot x^v = x \frac{1}{f(D)} \cdot v - \frac{f'(D)}{[f(D)]^2} \cdot v$$

v - will be only trigonometric

1. $(D^2+4)y = x \cdot \cos x$

$$P.I. = \frac{1}{(D^2+4)} \cdot x \cdot \cos x$$

$$= x \cdot \frac{1}{D^2+4} \cdot \cos x - \frac{2D}{(D^2+4)^2} \cdot \cos x$$

$$= x \cdot \frac{1}{-1+4} \cdot \cos x - \frac{-2 \cdot \sin x}{(-1+4)^2}$$

$$= \frac{x}{3} \cdot \cos x + \frac{2}{9} \cdot \sin x$$

2. $(D^2-2D+1)y = x e^x \sin x$

$$P.I. = \frac{1}{(D^2-2D+1)} \cdot x \cdot e^x \cdot \sin x$$

$$= e^x \left[x \frac{1}{(D-1)^2} \cdot \sin x - \frac{2D-2}{(D^2-2D+1)^2} \cdot \sin x \right]$$

$$= e^x \left[x \cdot \frac{1}{D^2-2D+1} \sin x - \frac{2 \cos x - 2 \sin x}{(D^2-2D+1)^2} \right]$$

or $= e^x \left[\frac{1}{(D+1-1)^2} x^2 \cdot \sin x \right]$

$$= e^x \left[\frac{1}{D^2} \cdot x^2 \cdot \sin x \right]$$

$$= e^x \left[x \frac{1}{D^2} \cdot \sin x - \frac{2D}{D^4} \cdot \sin x \right]$$

$$= e^x \left[x \frac{1}{-1} \sin x - \frac{2 \cos x}{(-1)^2} \right]$$

$$= e^x [-x \sin x - 2 \cos x]$$

Differential equation with variable coefficient

A D.E. is of the form

$$a_0 \cdot x^n \cdot \frac{d^n y}{dy^n} + a_1 \cdot x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 \cdot x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = Q$$

where, $a_0, a_1, \dots, a_n$ are constants & Q be the function of x , is said to be homogenous D.E. with variable constant or Cauchy-Euler form of D.E.

(Denominator function is independant variable)

$$x = e^z$$

$$z = \log x$$

$$\frac{dz}{dx} = \frac{1}{x}$$

$$\frac{d}{dz} = \frac{d}{dx} \frac{dx}{dz} = x \cdot \frac{d}{dx}$$

$$x \cdot \frac{d}{dx} = \frac{d}{dz} = \theta$$

$$D = \frac{d}{dx}$$

$$\theta = \frac{d}{dz}$$

Proof:

$$\frac{d}{dz} \left(x \cdot \frac{d}{dx} \right) = \frac{d^2}{dz^2}$$

$$\frac{d}{dx} \cdot x \cdot \frac{d}{dx} \cdot \frac{dx}{dz} = \theta^2$$

$$x \left[x \cdot \frac{d^2}{dx^2} + \frac{d}{dx} (1) \right] = \theta^2$$

$$x^2 \cdot \frac{d^2}{dx^2} + x \cdot \frac{d}{dx} = \theta^2$$

$$x^2 \cdot \frac{d^2}{dx^2} + 0 = 0^2$$

$$\frac{x^2 \cdot d^2}{dx^2} = 0^2 - 0 = 0(0-1)$$

similarly

$$\begin{aligned} \frac{x^3 d^3}{dx^3} &= 0(0-1)(0-2) \\ &= 0^3 - 3 \cdot 0^2 + 2 \cdot 0 \end{aligned}$$

$$xD = 0$$

$$x^2 D^2 = 0^2 - 0$$

$$x^3 D^3 = 0^3 - 3 \cdot 0^2 + 2 \cdot 0$$

$$1. (x^2 D^2 - xD - 3)y = 0$$

$$(0^2 - 0 - 0 - 3)y = 0$$

$$(0^2 - 2 \cdot 0 - 3)y = 0$$

$$m^2 - 2m - 3 = 0$$

$$(m-3)(m+1) = 0$$

$$m = 3, -1$$

solⁿ is

$$y = c_1 \cdot e^{3x} + c_2 \cdot e^{-x}$$

$$= c_1 \cdot x^3 + c_2$$

$$= c_1 \cdot (e^x)^3 + c_2 \cdot (e^x)^{-1}$$

$$y = c_1 \cdot x^3 + c_2 \cdot x^{-1}$$

$$2. (x^3 D^3 + 3x^2 D^2 - 2xD + 2)y = 0$$

$$0^3 - 3 \cdot 0^2 + 2 \cdot 0 + 3(0^2 - 0) - 2(0) + 2 = 0$$

$$0^3 - 3 \cdot 0^2 + 3 \cdot 0^2 - 3 \cdot 0 + 2 = 0$$

$$0^3 - 3 \cdot 0 + 2 = 0$$

$$m^3 - 3m + 2 = 0$$

$$\text{sum} = 1 - 3 + 2 = 0$$

$$\therefore m = 1 \text{ is root}$$

$m=1$	1	0	-3	2
	0	1	1	-2
	1	1	-2	0
$m=1$	0	1	2	
	1	2	0	

$$m+2=0$$

$$m=-2$$

solution is

$$y = (C_1 + C_2 D) e^z + C_3 e^{-2z}$$

$$= (C_1 + C_2 \log x) \cdot x + C_3 \cdot x^{-2}$$

$$3 \cdot (x^2 D^2 + x D - 4) y = 0 \quad y(0) = 0$$

$$(0^2 - 0 + 0 - 4) y = 0$$

$$(0^2 - 4) y = 0$$

$$m^2 - 4 = 0$$

$$m^2 = +4$$

$$m = \pm 2$$

$$y = C_1 e^{2z} + C_2 e^{-2z}$$

$$= C_1 x^{2x} + C_2 x^{-2}$$

$$y = C_1 x^2 + \frac{C_2}{x^2}$$

$$\text{At } y(0) = 0$$

$$0 = 0 + C_2$$

$$C_2 = 0$$

$$\text{At } y(1) = 1$$

$$1 = C_1 + C_2$$

$$C_1 = 1$$

$$y = x^2$$

$$4. x^2 y'' + xy' - y = 0$$

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$$(0^2 - 0 + 0 - 1)y = 0$$

$$(0^2 - 1)y = 0$$

$$m^2 = 1$$

$$m = \pm 1$$

$$y = c_1 \cdot e^x + c_2 \cdot e^{-x}$$

$$= c_1 x + c_2 \cdot \frac{1}{x}$$

$$5. x^2 \cdot \frac{d^3 y}{dx^3} + 2x \cdot \frac{d^2 y}{dx^2} - 2 \cdot \frac{dy}{dx} = 0$$

$$(x^2 D^3 + 2x D^2 - 2D)y = 0$$

$$(x^3 D^3 + 2x^2 D^2 - 2x D)y = 0$$

$$(0^3 - 3 \cdot 0^2 + 2 \cdot 0 - 2 \cdot 0)y = 0$$

$$(0^3 - 3 \cdot 0^2 + 2 \cdot 0 + 2 \cdot 0^2 - 2 \cdot 0 - 2 \cdot 0)y = 0$$

$$(0^3 - 0^2 - 2 \cdot 0)y = 0$$

$$m^3 - m^2 - 2m = 0$$

$$m(m^2 - m - 2) = 0$$

$$m(m - 2)(m + 1) = 0$$

$$m = 0, 2, -1$$

$$y = c_1 \cdot e^{0x} + c_2 \cdot e^{2x} + c_3 \cdot e^{-x}$$

$$= c_1 + c_2 \cdot x^2 + c_3 \cdot \frac{1}{x}$$

$$6. (4x^2 D^2 + 12x D + 3)y = 0$$

$$(4(0^2 - 0) + 12 \cdot 0 + 3)y = 0$$

$$(4 \cdot 0^2 - 4 \cdot 0 + 12 \cdot 0 + 3)y = 0$$

$$(4 \cdot 0^2 + 8 \cdot 0 + 3)y = 0$$

$$4m^2 + 8m + 3 = 0$$

$$(2m+1)(2m+3) = 0$$

$$m = -\frac{1}{2}, -\frac{3}{2}$$

$$y = C_1 x^{-1/2} + C_2 \cdot x^{-3/2}$$

Laplace's Transformation:

Let $f(t)$ be function in t defined for all positive values of t . then Laplace transform of $f(t)$ is denoted by $L[f(t)]$ and is defined as

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

then

$f(t) = L^{-1}[F(s)]$ is called inverse Laplace's transform of $F(s)$.

$$L(t^n) = \int_0^{\infty} e^{-st} t^n dt$$

let

$$x = st \Rightarrow t = \frac{x}{s}$$

$$dt = \frac{dx}{s}$$

$$\text{if } t=0, \quad x=0$$

$$t=\infty \quad x=\infty$$

$$= \int_0^{\infty} e^{-x} \left(\frac{x}{s}\right)^n \cdot \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^{\infty} e^{-x} x^n dx$$

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\Gamma(n+1) = n \Gamma(n) \quad \text{if } n > 0$$

$$\Gamma(n+1) = n! \quad \text{if } n \in \mathbb{N}$$

$$\Gamma = 1 \quad \Gamma_{1/2} = \sqrt{\pi}$$

$$L(t^n) = \frac{\sqrt{n+1}}{s^{n+1}}$$

$$= \frac{n!}{s^{n+1}} \quad \text{if } n \in \mathbb{N}$$

$$= \frac{n \sqrt{n}}{s^{n+1}} \quad \text{if } n > 0$$

$$L(1) = \frac{1}{s} \quad \text{if } n=0$$

$$L(t) = \frac{1}{s^2} \quad \text{if } n=1$$

$$L(t^2) = \frac{2}{s^3}$$

$$L(e^{at}) = \frac{1}{s-a}$$

$$L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(e^{-at}) = \frac{1}{s+a}$$

$$L(\sin hat) = \frac{a}{s^2-a^2}$$

$$L(\sin at) = \frac{a}{s^2+a^2}$$

$$L(\cosh at) = \frac{s}{s^2-a^2}$$

$$1. L(t^2+6t+8)$$

$$= L(t^2) + 6L(t) + 8L(1)$$

$$= \frac{2}{s^3} + \frac{6}{s^2} + \frac{8}{s}$$

$$2. L(\sqrt{t})$$

$$= L(t^{1/2})$$

$$= \frac{\sqrt{1/2+1}}{1/2+1}$$

$$= \frac{1/2 \sqrt{1/2}}{5^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2 \cdot 5^{3/2}}$$

$1/2$ - positive number.

$$3. L\left(\frac{1}{\sqrt{t}}\right)$$

$$= L(t^{-1/2})$$

$$= \frac{\Gamma(-1/2+1)}{s^{-1/2+1}}$$

$$= \frac{\Gamma(1/2)}{s^{1/2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$4. L(e^{2t} + 3e^{-2t})$$

$$= L(e^{2t}) + 3L(e^{-2t})$$

$$= \frac{1}{s-2} + 3 \frac{1}{s+2}$$

$$5. L(\sin^2 t)$$

$$= L\left(\frac{1 - \cos 2t}{2}\right)$$

$$= \frac{1}{2} [L(1) - L(\cos 2t)]$$

$$= \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2+4} \right]$$

$$= \frac{1}{2} \left[\frac{s^2+4-s^2}{s(s^2+4)} \right]$$

$$= \frac{2}{s(s^2+4)}$$

$$6. L(\cos^3 t)$$

$$= L\left[\frac{\cos 3t - 3 \cos t}{4}\right]$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$= \frac{1}{4} L(\cos 3t) - 3 L(\cos t)$$

$$= \frac{1}{4} \left[\frac{s}{s^2+9} + \frac{3 \times 1}{s^2+1} \right]$$

$$7. \quad L(\sin 2t \cdot \cos t)$$

$$= \frac{1}{2} L(2 \sin 2t \cos t)$$

$$= \frac{1}{2} L(\sin 3t + \sin t)$$

$$= \frac{1}{2} \left[\frac{3}{9+s^2} + \frac{1}{1+s^2} \right]$$

$$f(t) = 0 \quad 0 < t < 2$$

$$= 3 \quad t \geq 2$$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

$$= \int_0^2 e^{-st} \cdot 0 \cdot dt + \int_2^{\infty} e^{-st} \cdot 3 \cdot dt$$

$$= 0 + 3 \left(\frac{e^{-st}}{-s} \right)_2^{\infty}$$

$$= \frac{-3}{s} (e^{-\infty} - e^{-2s})$$

$$= \frac{-3}{s} (0 - e^{-2s})$$

$$= \frac{3}{s} e^{-2s}$$

Shifting theorem:

$$\text{If } L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

then,

$$L[e^{at} f(t)] = F(s-a)$$

L.H.S.

$$= \int_0^{\infty} e^{-st} \cdot e^{at} f(t) dt$$

$$= \int_0^{\infty} e^{-(s-a)t} f(t) dt$$

$$= F(s-a)$$

$$\therefore L(t^n) = \frac{n!}{s^{n+1}} \quad \text{if } n \in \mathbb{N}$$

$$L(e^{at} \cdot t^n) = \frac{n!}{(s-a)^{n+1}}$$

$$L(e^{-at} \cdot t^n) = \frac{n!}{(s+a)^{n+1}}$$

$$L(e^{at} \cdot \sin bt) = \frac{b}{(s-a)^2 + b^2}$$

$$\int L(e^{-at} \cdot \sin bt) = \frac{b}{(s+a)^2 + b^2}$$

$$L(e^{at} \cdot \cos bt) = \frac{s-a}{(s-a)^2 + b^2}$$

$$L(e^{-at} \cdot \cos bt) = \frac{s+a}{(s+a)^2 + b^2}$$

$$L(e^{at} \cdot \sinh bt) = \frac{b}{(s-a)^2 - b^2}$$

$$L(e^{-at} \cdot \cosh bt) = \frac{s-a}{(s-a)^2 - b^2}$$

e.g. $L(t \cdot \sin at) \Rightarrow$

We know $e^{ait} = \cos at + i \sin at$
imaginary part

$$= \text{Imaginary part of } L(t \cdot e^{ait}) \quad L(t) = \frac{1}{s^2}$$

$$= \text{Imaginary part of } \left[\frac{1}{(s-a)^2} \times \frac{(s+ai)^2}{(s+ai)^2} \right]$$

$$= \text{I.P. of } \left[\frac{s^2 - a^2 + 2ias}{(s^2 + a^2)^2} \right]$$

$$* L(t \cdot \sin at) = \frac{2as}{(s^2 + a^2)^2}$$

$$* L(t \cdot \cos at) = \text{Real part of } \left[\frac{s^2 - a^2 + 2ias}{(s^2 + a^2)^2} \right]$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

Inverse Laplace Transform:

(100)

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L\left(\frac{t^n}{n!}\right) = \frac{1}{s^{n+1}}$$

$$L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$$

$$L^{-1}\left(\frac{1}{s}\right) = 1$$

$$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a} \sin at$$

$$L^{-1}\left(\frac{1}{s^2}\right) = t$$

$$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

$$L^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2}$$

$$L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a} \sinh at$$

$$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$$

$$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

Find: $L^{-1}\left(\frac{1}{2s-3}\right)$

$$= \frac{1}{2} L^{-1}\left(\frac{1}{s-3/2}\right)$$

$$= \frac{1}{2} \cdot e^{3t/2}$$

$L^{-1}\left(\frac{1}{s(s+1)}\right)$

$$= L^{-1}\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$$= L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$$= 1 - e^{-t}$$

$L^{-1}\left(\frac{1}{(s+1)^2}\right)$

$$= e^{-st} \cdot L^{-1}\left(\frac{1}{s^2}\right)$$

shifting theorem

$$= e^{-st} \cdot t$$

$$\mathcal{L}^{-1} \left(\frac{1}{(s+a)^2} \right)$$

$$= \mathcal{L}^{-1} \left(\frac{s+a-a}{(s+a)^2} \right)$$

$$= e^{-at} \cdot \mathcal{L}^{-1} \left(\frac{s}{s^2} - \frac{a}{s^2} \right)$$

$$= e^{-at} \cdot \mathcal{L}^{-1} \left(\frac{1}{s} - \frac{a}{s^2} \right) \quad \text{— shifting thm. of denominator}$$

$$= e^{-at} (1 - at)$$

when denominator is perfect square, arrange numerator in terms

$$\mathcal{L}^{-1} \left(\frac{s+23}{s^2+13-4s} \right)$$

↖ not perfect square - arrange as perfect square
(don't go for partial factor)

$$= \mathcal{L}^{-1} \left(\frac{s+23}{s^2-2 \cdot s \cdot 2 + 4+9} \right)$$

$$= \mathcal{L}^{-1} \left(\frac{s-2+25}{(s-2)^2+3^2} \right)$$

$$= e^{2t} \mathcal{L}^{-1} \left(\frac{s}{s^2+3^2} + 25 \frac{1}{s^2+3^2} \right)$$

$$= e^{2t} \left(\cos 3t + \frac{25}{3} \sin 3t \right)$$

$$\mathcal{L}^{-1} \left(\frac{s-2}{s^2+2s+2} \right)$$

$$= \mathcal{L}^{-1} \left(\frac{s-2}{s^2+2s+1+1} \right)$$

$$= \mathcal{L}^{-1} \left(\frac{s+1-3}{(s+1)^2+1} \right)$$

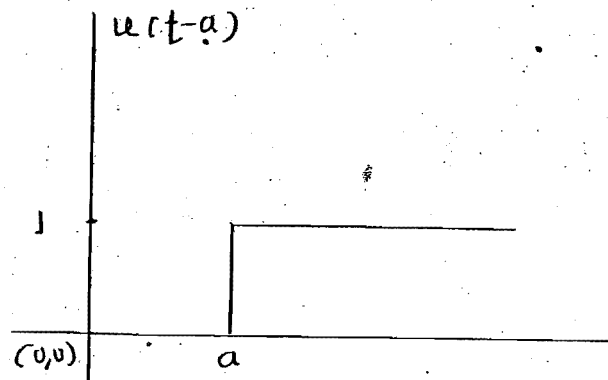
$$= e^{-t} \cdot \mathcal{L}^{-1} \left(\frac{s+1-3}{s^2+1} \right)$$

$$= e^{-t} \cdot \mathcal{L}^{-1} \left(\frac{s}{s^2+1} - \frac{2}{s^2+1} \right)$$

$$= e^{-t} (\cos t - 3 \sin t)$$

(10)

Unit step function:



$$u(t-a) = 0 \quad 0 < t < a$$

$$= 1 \quad t \geq a$$

$$L[u(t-a)] = \int_0^{\infty} e^{-st} \cdot u(t-a) \cdot dt$$

$$= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_a^{\infty} e^{-st} \cdot 1 \cdot dt$$

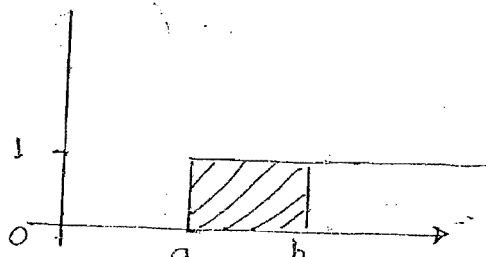
$$= 0 + \left(\frac{e^{-st}}{-s} \right)_a^{\infty}$$

$$= \frac{e^{-\infty} - e^{-as}}{-s}$$

$$= \frac{0 - e^{-as}}{-s}$$

$$= \frac{e^{-as}}{s}$$

Find Laplace's transform of function (shaded)



$$L(f(t)) = \frac{1}{s} e^{-as} - \frac{1}{s} e^{-bs}$$

$$= \frac{1}{s} (e^{-as} - e^{-bs})$$

periodic function:

A function is $f(t)$ said to be periodic function with a period T .

upper finite bound of the function is always period of a function ($T=4$)

$$\begin{aligned}L[f(t)] &= \frac{1}{1-e^{-4s}} \left[\int_0^2 e^{-st} (3) \cdot dt + \int_2^4 e^{-st} (0) \cdot dt \right] \\&= \frac{3}{1-e^{-4s}} \left[\frac{e^{-st}}{-s} \right]_0^2 + 0 \\&= \frac{3}{1-e^{-4s}} \left[\frac{e^0 - e^{-2s}}{s} \right] \quad (\text{lower limit put } s^{\text{th}}) \\&= \frac{3(1-e^{-2s})}{s(1-e^{-2s})(1+e^{-2s})} \\&= \frac{3}{s(1+e^{-2s})}\end{aligned}$$

$$f(t) = 1 \quad 0 < t < b$$

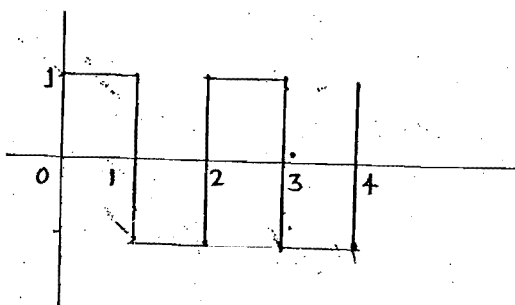
$$= -1 \quad b < t < 2b$$

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$$f(t+2b) = f(t)$$

$$\begin{aligned} \mathcal{L}[f(t)] &= \frac{1}{1 - e^{-2bs}} \left[\int_0^b e^{-st} (1) dt + \int_b^{2b} e^{-st} (-1) dt \right] \\ &= \frac{1}{1 - e^{-2bs}} \left[\left. \frac{e^{-st}}{-s} \right|_0^b + \left. \frac{e^{-st}}{s} \right|_b^{2b} \right] \\ &= \frac{1}{1 - e^{-2bs}} \left[\frac{1 - e^{-bs} + e^{-2bs} - e^{-bs}}{s} \right] \\ &= \frac{1}{1 - e^{-2bs}} \left[\frac{1 - 2e^{-bs} + e^{-2bs}}{s} \right] \\ &= \frac{(1 - e^{-bs})^2}{(1 - e^{-bs})(1 + e^{-bs})s} \\ &= \frac{(1 - e^{-bs})}{s(1 + e^{-bs})} \quad * \text{ (shortcut)} \end{aligned}$$

Q. A function $f(t)$ is defined in the graph find Laplace's transform of the function.



$$f(t) = 1 \quad 0 < t < 1$$

$$= -1 \quad 1 < t < 2$$

$$\mathcal{L}[f(t)] = \frac{1 - e^{-2}}{s(1 + e^{-1})}$$

If $L[f(t)] = F(s) = \frac{5s^2 + 23s + 6}{s(s^2 + 2s + 2)}$

then,

$$\lim_{t \rightarrow \infty} f(t) = ?$$

(a) 3

(b) 5

(c) $\frac{17}{2}$

(d) ∞

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s \cdot F(s)$$

$$\therefore \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s \left[\frac{5s^2 + 23s + 6}{s(s^2 + 2s + 2)} \right]$$

$$= \frac{5(0) + 23(0) + 6}{0 + 2(0) + 2}$$

$$= 3$$

If $L(f(t)) = \frac{3s+1}{s^3+4s^2+(k-3)s}$ and $\lim_{t \rightarrow \infty} f(t) = 1$

then $k = ?$

$$\lim_{s \rightarrow 0} s \cdot f(s) = 1$$

$$= \lim_{s \rightarrow 0} s \left[\frac{3s+1}{s(s^2+4s+k-3)} \right]$$

$$1 = \frac{1}{0+0+k-3}$$

$$k-3 = 1$$

$$k = 4$$

If $L[f(t)] = \frac{1}{s^2(s+1)}$ then $f(t) = ?$

(a) $1 - 1 + e^{-t}$

(b) $t + 1 + e^{-t}$

(c) $-1 + e^{-t}$

(d) $2t + e^t$

for (a)

$$L(1) = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}$$

$$= \frac{(s+1) - (s+1)s + s^2}{s^2(s+1)}$$

$$= \frac{1}{s^2(s+1)}$$

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Complex Variables:

$$Z = x + iy$$

$$\bar{Z} = x - iy$$

$$Z \cdot \bar{Z} = |Z|^2 = x^2 + y^2$$

$Z = x + iy = r(\cos \theta + i \sin \theta)$ is called modulus-amplitude form or polar form

$$\text{mod. } Z = r = \sqrt{x^2 + y^2}$$

$$\text{amp. } Z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

$$1 = \cos \theta + i \sin \theta = e^{i0}$$

$$i = \cos \pi/2 + i \sin \pi/2 = e^{i\pi/2}$$

$$(1+i) = \sqrt{2} \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right]$$

$$= \sqrt{2} \cdot e^{i\pi/4}$$

$$i^i = (e^{i\pi/2})^i = e^{i^2 \pi/2} = e^{-\pi/2}$$

$$i = \sqrt{-1}$$

$$i^2 = -1$$

$$1. \frac{1+i}{1+j}$$

$$\begin{aligned}
 &= \frac{(1+3)(1-i)}{(1+1)(1-i)} \\
 &= \frac{j^2 - i + 3i - 3}{j^2 - 1} \\
 &= \frac{-1 + 2i - 3}{-2} \\
 &= \frac{-4 + 2i}{-2} \\
 &= \frac{-2(2-i)}{-2} \\
 &= (2-i)
 \end{aligned}$$

$$2. \frac{j+2i}{i-2}$$

$$\begin{aligned}
 &= \frac{(1+2i)(j+2)}{(i-2)(i+2)} \\
 &= \frac{i+2+2i^2+4i}{i^2-4} \\
 &= \frac{-2+5i+2}{-5} \\
 &= -i
 \end{aligned}$$

$$5. \frac{-5+10i}{3+4i}$$

$$\begin{aligned}
 &= \frac{(-5+10i)(3-4i)}{9-16i^2} \\
 &= \frac{-15+20i+30i-40i^2}{9-16i^2} \\
 &= \frac{25+50i}{25} \\
 &= 1+2i
 \end{aligned}$$

4. $\left| \frac{3+4i}{1-2i} \right|$

$$= \frac{|3+4i|}{|1-2i|}$$

$$= \frac{\sqrt{9+16}}{\sqrt{1+4}}$$

$$= \frac{5}{\sqrt{5}}$$

$$= \sqrt{5}$$

5. $(1+i)^8$

$$(1+i)^8 = [(1+i)^2]^4$$

$$= [1+2i+i^2]^4$$

$$= [1+2i-1]^4$$

$$= 16i^4$$

$$= 16$$

$$i^4 = (-1)(-1) = 1$$

6. The polar form of $2+2i$ is

a) $\frac{\pi}{4} e^{i\sqrt{2}}$

b) $\frac{\pi}{4} e^{i2\sqrt{2}}$

c) $\sqrt{2} \cdot e^{i\pi/4}$

d) $2\sqrt{2} \cdot e^{i\pi/4}$

$$r = \sqrt{4+4}$$

$$= 2\sqrt{2}$$

$$\text{polar form} = 2\sqrt{2} \cdot e^{i\pi/4}$$

Analytic function:

A single valued function which is defined and differential at each point of domain D is said to be an analytical function, in the domain.

Note:

The necessary and sufficient condition for a function $f(z) = u(x, y) + i \cdot v(x, y)$ to be analytic is, it should satisfy Cauchy-Riemann's (C-R) equation as.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Harmonic function:

Any function of x and y satisfying the Laplace's equation is said to be harmonic function.

$$\therefore u = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial y^2} = -2$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= 2 - 2$$

$$\nabla^2 u = 0$$

Note:

The real and imaginary parts of an analytic function satisfies the Laplace's equation.

$$\nabla^2 u = 0$$

$$\nabla^2 v = 0$$

Construction of analytic function:

$$\text{If } u = f(x, y)$$

shortcut*

$$f'(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ci$$

e.g. $u = x^2 + y^2$

$$\begin{aligned} u\left(\frac{z}{2}, \frac{z}{2i}\right) &= \frac{z^2}{4} - \frac{z^2}{4i^2} \\ &= \frac{z^2}{4} + \frac{z^2}{4} \\ &= \frac{z^2}{2} \end{aligned}$$

$$\begin{aligned} f(z) &= 2\left(\frac{z^2}{2}\right) - 0 + ci \\ &= z^2 + ci \end{aligned}$$

1. $u = x^3 - 3xy^2 + 3x + 1$

$$\begin{aligned} u\left(\frac{z}{2}, \frac{z}{2i}\right) &= \left(\frac{z}{2}\right)^3 - 3\left(\frac{z}{2}\right)\left(\frac{z^2}{4i^2}\right) + 3\left(\frac{z}{2}\right) + 1 \\ &= \frac{z^3}{8} + \frac{3z^3}{8} + \frac{3z}{2} + 1 \\ &= \frac{4z^3}{8} + \frac{3z}{2} + 1 \\ &= \frac{z^3}{2} + \frac{3z}{2} + 1 \end{aligned}$$

$$\begin{aligned} f(z) &= 2\left(\frac{z^3}{2} + \frac{3z}{2} + 1\right) - 1 + ci \\ &= z^3 + 3z + 2 - 1 + ci \\ &= z^3 + 3z + 1 + ci \end{aligned}$$

To find v (imaginary part)

$$\begin{aligned} dz &= \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy && \text{Total derivative} \\ dv &= \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy \end{aligned}$$

$$dv = \frac{-\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \text{which is exact}$$

$$u = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\int dv = \int 2y \cdot dx + \int 2x \cdot dy$$

$$v = 2xy$$

terms not containing x

$$2. u = e^x \cdot \cos y$$

$$\frac{\partial u}{\partial x} = e^x \cdot \cos y \quad \frac{\partial u}{\partial y} = e^x (-\sin y)$$

$$v = \int e^x \cdot \sin y \, dx + \int e^x \cdot \cos y \, dy$$

$$v = e^x \cdot \sin y$$

$$f(z) = u + iv$$

$$= e^x \cdot \cos y + (e^x \sin y) i$$

$$= e^x (\cos y + i \sin y)$$

$$= e^x \cdot e^{iy}$$

$$= e^{x+iy} = e^z$$

$$3. v = xy \quad \text{find } u$$

$$\frac{\partial u}{\partial x} = y \quad \frac{\partial u}{\partial y} = x$$

$$v = \int x \cdot dy$$

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy$$

$$= \frac{\partial v}{\partial y} \cdot dx - \frac{\partial v}{\partial x} \cdot dy$$

$$\begin{aligned}\int du &= \int x \cdot dx - \int y \cdot dy \\ &= \frac{x^2}{2} - \frac{y^2}{2} \\ &= \frac{1}{2} (x^2 - y^2)\end{aligned}$$

Singular points:

The singular points are those at which the given function $f(z)$ is not analytic. These points are also called singularities or poles.

Q. Find the singular points of function. $f(z) = \frac{1-2z}{z(z-1)(z-2)}$

singular points = $z = 0, 1, 2$.

all the poles above are of order 1, (simple poles)

$$f(z) = \frac{z^2}{(z-1)(z+1)(z-i)(z+i)} = \frac{z^2}{z^4-1} = \frac{z^2}{(z^2-1)(z^2+1)}$$

poles = $\pm 1, \pm i$ of order 1

$$f(z) = \frac{z^2}{z^2-3z+2}$$

$$z^2-3z+2=0$$

$$(z-1)(z-2)=0$$

$z = 1, 2$, are poles

$$f(z) = \frac{z^2}{(z-1)^2(z-2)^3}$$

poles, $z = 1$, of order 2

$z = 2$ of order 3

Cauchy's theorem:

If $f(z)$ is an analytic function and its derivative $f'(z)$ is continuous at all points inside and on a simple and closed curve C , then:

$$\int_C f(z) \cdot dz = 0$$

(pole is outside the region)

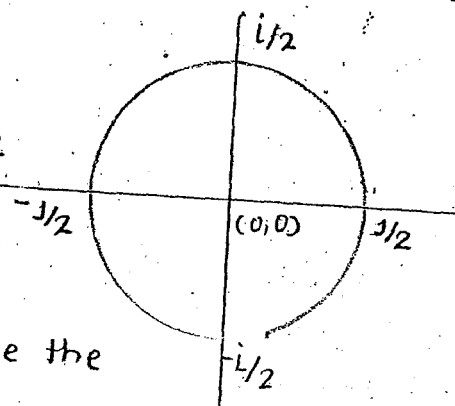
Q. The value of $\int_C \frac{3z^2 + 7z + 1}{z+1} dz$ where C is circle, $|z| = 1/2$

$$|z| = 1/2$$

$$(x^2 + y^2)^2 = (1/2)^2, r^{-1}$$

centre $(0,0)$ & radius $1/2$

pole, $z = -1$ i.e. outside the circle



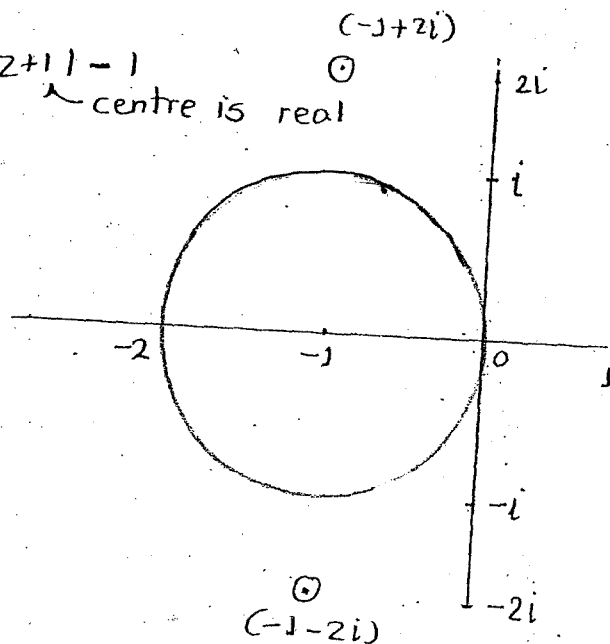
$$\int_C \frac{3z^2 + 7z + 1}{z+1} dz = 0$$

Q. $\int_C \frac{z+4}{z^2+2z+5} dz$ where C is $|z+1| = 1$

$$z^2 + 2z + 5 = 0$$

$$z = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= -1 \pm 2i$$



$$5. \oint \frac{-3z+4}{z^2+4z+5} dz = ? \quad |z|=1$$

$$z^2+4z+5=0$$

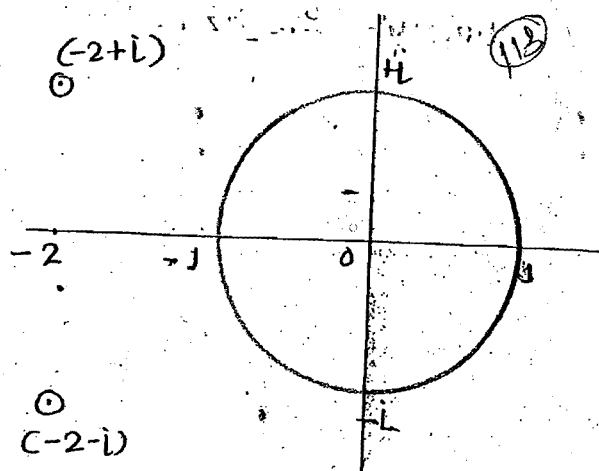
$$z = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$\text{poles} = -2 \pm i$$

poles are outside circle.

$$\oint \frac{-3z+4}{z^2+4z+5} dz = 0$$



2014 GATE

Cauchy's integral formula:

If $f(z)$ is an analytic function inside and on a simple and closed curve C , then, if 'a' is any point inside C then,

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i \cdot f(a)$$

$$\oint_C \frac{f(z)}{(z-a)^2} dz = 2\pi i f'(a)$$

$$\oint_C \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i}{2!} f''(a)$$

$$\oint_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$$

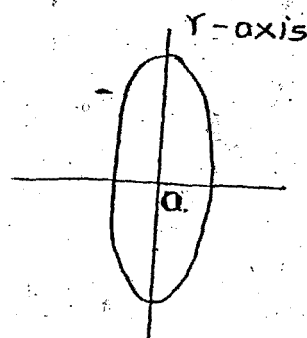
$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Q Find $\oint_c \frac{\cos z}{z} dz$ where c is ellipse $9x^2 + 4y^2 = 1$.

$$\frac{x^2}{(1/9)} + \frac{y^2}{(1/4)} = 1$$

More. $\rightarrow$

$$\begin{aligned} \oint_c \frac{\cos z}{z} dz &= 2\pi i [\cos z]_{z=0} \\ &= 2\pi i \cdot \cos 0 \\ &= 2\pi i \end{aligned}$$



Q The value of $\oint_c \frac{1}{z^2-1}$ where c is circle of $x^2 + y^2 = 4$

$$z^2 - 1 = 0$$

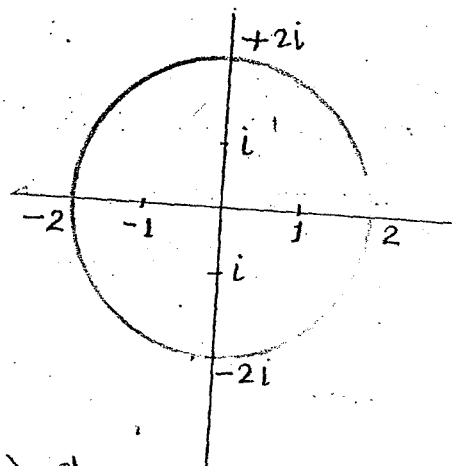
$$(z-1)(z+1) = 0$$

poles, $z = 1, -1$

both are inside the circle.

Method 1 (partial fraction) $\therefore$

$$\begin{aligned} \oint_c \frac{1}{z^2-1} &= \oint_c \left(\frac{1}{z-1} - \frac{1}{z+1} \right) \frac{dz}{2} \\ &= \frac{1}{2} \oint_c \left(\frac{1}{z-1} - \frac{1}{z+1} \right) dz \\ &= \frac{1}{2} [2\pi i (1) - 2\pi i (1)] \\ &= 0 \end{aligned}$$



Method 2.

$(z-1)$ take as pole $\&$ $(z+1)$ - as function
 $(z+1)$ - as function $\&$ $(z-1)$ - as pole.

$$\oint_c \frac{1}{z^2-1} = \int_c \frac{1}{(z+1)(z-1)} dz = \int_c \frac{1}{z+1} dz + \int_c \frac{1}{(z-1)(z+1)} dz$$

$$= 2\pi i \left(\frac{1}{z+1} \right)_{z=-1} + 2\pi i \left(\frac{1}{z-1} \right)_{z=-1}$$

$$= 2\pi i \left(\frac{1}{2} \right) + 2\pi i \left(\frac{-1}{2} \right)$$

$$= 0$$

(114)

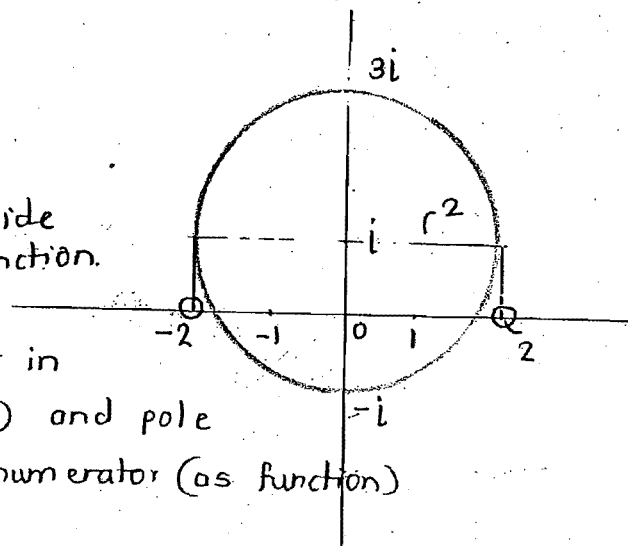
Q. Find $\oint_C \frac{z-1}{(z+1)^2(z-2)} dz$ where C is circle $|z-i|=2$

$$(z+1)^2(z-2) = 0$$

$$z = -1 \text{ and } z = 2$$

inside the function

outside function.



* pole inside function, put in denominator (as pole) and pole outside function put in numerator (as function)

$$\oint_C \frac{\left(\frac{z-1}{z-2} \right)}{(z+1)^2} dz$$

$$f(z) = \frac{z-1}{z-2} = 1 + \frac{1}{z-2}$$

$$f'(z) = 0 - \frac{1}{(z-2)^2}$$

$$\therefore \oint_C \frac{\left(\frac{z-1}{z-2} \right)}{(z+1)^2} dz = 2\pi i \left(\frac{-1}{(z-2)^2} \right)_{z=-1}$$

$$= \frac{-2\pi i}{(-1-2)^2}$$

$$= \frac{-2\pi i}{9}$$

Q. For $|z+1|=2$ find $\oint_C \frac{z^2-4}{z^2+4} dz$

$$z^2+4=0$$

$$(z+2i)(z-2i)=0$$

$$z = 2i, -2i$$

inside outside

$$\begin{aligned} \oint_C \frac{\left(\frac{z^2-4}{z+2i}\right)}{z-2i} &= 2\pi i \left(\frac{z^2-4}{z+2i}\right)_{z=2i} \\ &= 2\pi i \left(\frac{-4-4}{2i+2i}\right) \\ &= \frac{-8\pi}{2} \end{aligned}$$

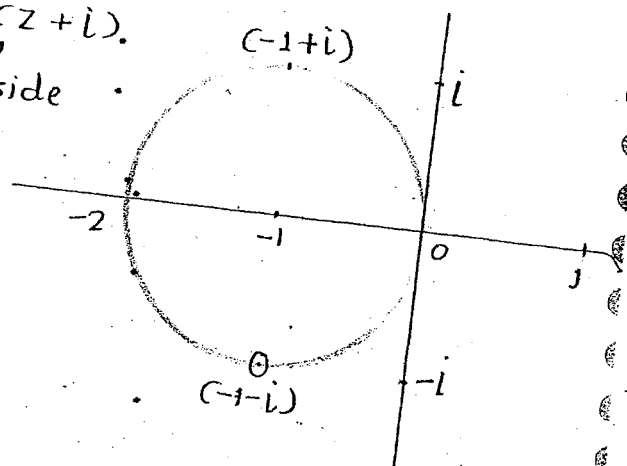
~~Residue~~
 ~~$\frac{-8\pi}{2} = -4\pi$~~

Q. Find $\oint_C \frac{z^2}{z^4-1} dz$ where C is the circle $|z+1|=1$

$$z^4-1 = (z-1)(z+1)(z-i)(z+i)$$

outside inside outside

$$\begin{aligned} \oint_C \frac{z^2}{z^4-1} &= \oint_C \frac{z^2}{(z-1)(z^2-1)} \\ &= \oint_C \frac{z^2}{(z-1)(z^2+1)} \end{aligned}$$



$$= 2\pi i \left[\frac{z^2}{(z-1)(z^2+1)} \right]_{z=-1}$$

$$= 2\pi i \left(\frac{1}{-2 \times 2} \right)$$

$$= -\frac{\pi i}{2}$$

Residue at $z=a$ where 'a' is the pole of order 'n'. (not asked)

$$\lim_{z \rightarrow a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n \cdot f(z) \right]$$

Q. Find residue of function at respective poles

$$f(z) = \frac{1-2z}{z(z-1)(z-2)}$$

poles of function = 0, 1, 2 (of first order, $n=1$)

At $z=0$

$$\begin{aligned} \text{Residue} &= \lim_{z \rightarrow 0} \frac{z \cdot (1-2z)}{z(z-1)(z-2)} \\ &= \lim_{z \rightarrow 0} \frac{(1-2z)}{(z-1)(z-2)} = \frac{1}{(-1)(-2)} = \frac{1}{2} \end{aligned}$$

At $z=1$

$$\begin{aligned} &= \lim_{z \rightarrow 1} \frac{(1-1) \cdot (1-2z)}{z(z-1)(z-2)} \\ &= \frac{-1}{-1} = 1 \end{aligned}$$

At $z=2$

$$= \lim_{z \rightarrow 2} \frac{(z-2) \cdot (1-2z)}{z(z-1)(z-2)}$$

Q. Find residue

$$f(z) = \frac{1}{z(z+2)^3}$$

At $z=-2$, $n=3$

$$\begin{aligned} \text{Residue} &= \lim_{z \rightarrow -2} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z+2)^3 \cdot \frac{1}{z(z+2)^3} \right] \\ &= \lim_{z \rightarrow -2} \frac{1}{2!} \left(-\frac{2}{z^3} \right) = \frac{1}{(-2)^3} = -\frac{1}{8} \end{aligned}$$

Numerical methods:

Solution of non-linear equations:

Let $f(x)=0$ be any non-linear equation. To find the approximate root of $f(x)=0$, choose any two values of $x=a, b$ particularly adjacent values in such a way that these functional values must have opposite signs. Then we must say that root lies between A and b . This approximate root can be found by

- (i) Method of bisection
- (ii) Regula-falsi method
- (iii) Newton-Raphson method

(i) Method of bisection

$$x_1 = \frac{a+b}{2}$$

$$\text{if } f(x_1) < 0$$

$$f(a) \cdot f(x_1) < 0$$

$$x_2 = \frac{x_1 + b}{2}$$

$$f(x_2) > 0$$

$$x_3 = \frac{x_1 + x_2}{2}$$

(must go upto 13th to 14th approximation)

The rate of convergence in bisection method is very-very slow. Comparatively bisection method, the Regula-Falsi method has little fast rate of convergence.

(ii) Regula-falsi method:

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

(must go upto 7th to 8th approximation)

(i) Newton-Raphson method:

(116)

It has very fast rate of convergence. The convergence of Newton-Raphson method is quadratic convergence or the 2nd order convergence.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

1. $f(x) = x^3 + 2x - 5 = 0$ $x_0 = 1$ then $x_1 = ?$

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 1 - \frac{(1)^3 + 2(1) - 5}{-3(1)^2 + 2} \\ &= 1 - \frac{-2}{5} \\ &= 1.4 \end{aligned}$$

2. $f(x) = x^3 - 2x - 5 = 0$, $x_1 = ?$

$$f(0) = -5 < 0$$

$$\begin{aligned} f(1) &= 1 - 2 - 5 \\ &= -6 < 0 \end{aligned}$$

$$\begin{aligned} f(2) &= 8 - 4 - 5 \\ &= -1 < 0 \end{aligned}$$

$$\begin{aligned} f(3) &= 27 - 6 - 5 \\ &= 16 > 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} f(3) &= 27 - 6 - 5 \\ &= 16 > 0 \end{aligned}} \right\} \therefore x_0 = 2 \text{ - nearer to 0}$$

$$x_1 = 2 - \frac{(2)^3 - 2(2) - 5}{3(2)^2 - 2}$$

$$x_1 = 2.1$$

3. Starting from $x_0 = 1$, $f(x) = x^3 + 3x - 7 = 0$, - find x_1

$$x_1 = 1 - \frac{3(1)^3 + 3(1) - 7}{3(1) + 3}$$

$$= 1 - \frac{-3}{6}$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

1.5

4. $f(x) = x^2 - 2x - 1$, $x_0 = 2$, $x_1 = ?$

$$x_1 = 2 - \frac{(2)^2 - 2(2) - 1}{2(2) - 2}$$

$$= 2 - \frac{-1}{2}$$

$$= \frac{5}{2} = 2.5$$

$$x_2 = 2.5 - \frac{(2.5)^2 - 2(2.5) - 1}{2(2.5) - 2}$$

$$= 2.5 - \frac{6.25 - 5 - 1}{5 - 2}$$

$$= 2.5 - \frac{0.25}{3}$$

$$= \frac{7.25}{3} = 2.417$$

5. $f(x) = e^x - 1 = 0$, $x_0 = -1$

$$x_1 = -1 - \frac{e^{-1} - 1}{e^{-1}}$$

$$= -1 - \frac{(1 - e) \cdot e}{e}$$

$$= -1 + 1 + e$$

$$= e - 2$$

$$= 2.71 - 2$$

$$= 0.71$$

Q. The Newton-Raphson iteration $x_{n+1} = \frac{1}{2} \left(x_n + \frac{R}{x_n} \right)$ can be used to compute. (18)

(a) square of R

(b) square root of R

(c) $\log R$

(d) R^{-1}

$\therefore$ value of $x_{n+1} = x_n = \alpha$ we stop iterations

$$\alpha = \frac{1}{2} \left(\alpha + \frac{R}{\alpha} \right)$$

$$2\alpha - \alpha = \frac{R}{\alpha}$$

$$\alpha^2 = R$$

$$\alpha = \sqrt{R}$$

used to find square root of R.

The value $x_{n+1} = \frac{1}{2} \left(x_n + \frac{5}{x_n} \right)$ converges to ..

$$\text{Ans} = \sqrt{5}$$

Q. Recursion relation:

$$f(x) = x - e^{-x},$$

It can be solved by Newton-Raphson method

(a) $x_{n+1} = e^{-x_n}$

(b) $x_{n+1} = x_n - e^{-x_n}$

✓ (c) $x_{n+1} = \frac{(1+x_n) \cdot e^{-x_n}}{1+e^{-x_n}}$

(d) $x_{n+1} = \frac{x_n^2 - e^{-x_n}(1+x_n) - 1}{x_n - e^{-x_n}}$

Ans. $f'(x) = 1 + e^{-x}$

Consider the series $x_{n+1} = \frac{x_n}{2} + \frac{9}{8x_n+1}$ & $x_0 = 0.5$ is obtained by Newton's-Raphson method, then series converges to.

(a) 1.4

(b) 1.414

(c) 1.5

(d) 1.6

$$x_{n+1} = x_n = \alpha$$

$$\alpha = \frac{\alpha}{2} + \frac{9}{8\alpha+1}$$

$$= \frac{8\alpha^2 + 18}{16\alpha + 2}$$

$$16\alpha^2 + 2\alpha = 8\alpha^2 + 18$$

$$8\alpha^2 = 18 - 2\alpha$$

$$\alpha^2 = \frac{9}{4}$$

$$\alpha = \sqrt{\frac{9}{4}} = 1.5$$

The solution of variables x_1 and x_2 for the equations

$$u = 10x_2 \sin x_1 - 0.8 = 0$$

$$v = 10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0 \text{ is to be solved}$$

by Newton-Raphson method assuming initial values of $x_1 = 0.0$ and $x_2 = 1.0$ the $J(x_1, x_2)$ is

$$J(x_1, x_2) = \frac{\partial(u, v)}{\partial(x_1, x_2)} = \begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial u}{\partial x_1} = 10x_2 \cos x_1$$

$$\frac{\partial u}{\partial x_2} = 10 \sin x_1$$

$$\frac{\partial v}{\partial x_1} = 10x_2 \sin x_1$$

$$\frac{\partial v}{\partial x_2} = 20x_2 - 10 \cos x_1$$

At $x_1 = 0, x_2 = 1.$

$$J(x_1, x_2) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

Q. The Newton Raphson method is used to solve the equation. (18)

$f(x) = -(x-1)^2 + x - 3 = 0$, the method will fail in very first iteration if the initial guess is ?

$$f(x) = x^2 - 2x + 1 + x - 3$$

$$= x^2 - x - 2$$

$$f'(x) = 2x - 1$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - x_0 - 2}{2x_0 - 1}$$

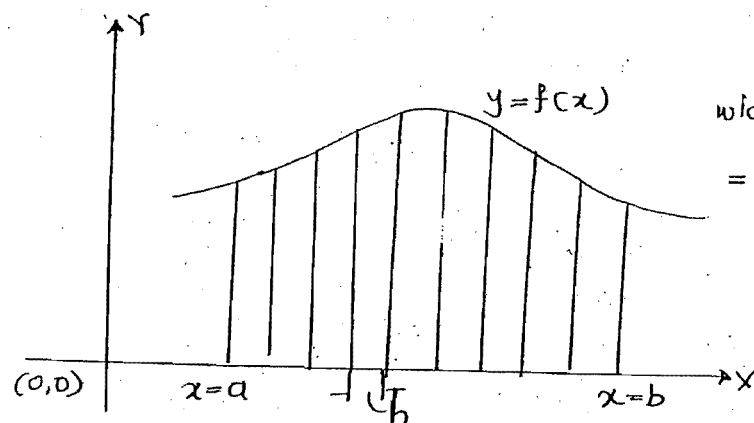
Method will fail if denominator becomes zero

$$2x_0 - 1 = 0$$

$$2x_0 = 1$$

$$x_0 = 1/2 = 0.5$$

Numerical integration :



width of each in $-vo$

$$= h = \frac{b-a}{n}$$

Trapezoidal rule:

$$\int_a^b f(x) dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \dots + y_{n-1})]$$

Simpson's $\frac{1}{3}$ rule:

$$\int_a^b f(x) dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots) + 2(y_2 + y_4 + \dots)]$$

The accuracy of trapezoidal rule or the order of integration of trapezoidal rule is $O(h^2)$ and accuracy of the Simpson's rule ($1/3$) is $O(h^4)$.

Q. The value of integral $\int_0^1 \frac{1}{1+x^2} dx$ by trapezoidal and the Simpson's rule by dividing interval into 4 equal parts. & hence find value of π .

$$y = \frac{1}{1+x^2} \quad h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

x	0	0.25	0.50	0.75	1.0
y	1	0.9412	0.8	0.6400	0.5

Trapezoidal rule,

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{0.25}{2} [(1+0.5) + 2(0.9412 + 0.8 + 0.64)]$$

$$(\tan^{-1}x)_0^1 = 0.7828$$

$$\tan^{-1}1 - \tan^{-1}0 = 0.7828$$

$$\frac{\pi}{4} = 0.7828$$

$$\pi = 3.1312$$

$$\text{Actual } \pi = 3.1415$$

(difference in 2nd decimal
i.e. $O(h^2)$)

Simpson's rule:

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{0.25}{3} [(1+0.5) + 4(0.9412 + 0.64) + 2(0.8)]$$

$$\frac{\pi}{4} = 0.7854$$

$$\pi = 3.1416$$

$$\text{Actual } \pi = 3.1415$$

(difference in 4th decimal
i.e. $O(h^4)$)

Q. The value of $\int_0^6 \frac{1}{1+x^2} \cdot dx$ by trapezoidal rule by dividing (119)
the interval into 6 equal parts is?

- (a) 1.4132 (b) 1.3756 (c) 1.2326 (d) 1.0987

$$\begin{aligned} \int_0^6 \frac{1}{1+x^2} &= (\tan^{-1}x)_0^6 \\ &= \tan^{-1}6 - \tan^{-1}0 \\ &= 1.4056^\circ \end{aligned}$$

(if there is no comparison between the options, then only integrate)

Q. The value of $\int_0^{\pi/4} \frac{\sin x}{\cos^3 x} \cdot dx$ by Simpson's rule. ($n=4$) is

- (a) 0.4344 (b) 0.3787 (c) 0.5044 (d) 0.5529

$$\begin{aligned} &\int_0^{\pi/4} \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^2 x} dx \\ &= \int_0^{\pi/4} \tan x \cdot \sec^2 x \cdot dx \\ &= \left(\frac{\tan^2 x}{2} \right)_0^{\pi/4} \\ &= \frac{1}{2} (1) = 0.5 \end{aligned}$$

Q. $\int_1^3 \log_e x \cdot dx$ by Simpson's rule. ($n=2$)

- (a) 0.5 (b) 0.8 (c) 1.00 (d) 1.29

$$\begin{aligned} \int_1^3 \log_e x \cdot dx &= [x \cdot \log x - x]_1^3 \\ &= (3 \log 3 - 3) - (0) \\ &= 3 \log 3 - 3 + 1 \\ &= 1.294 \end{aligned}$$

$$\int_1^3 \frac{1}{x} \cdot dx = ? \quad (n=2)$$

(a) 1.000

(b) 1.098

(c) 1.111

(d) 1.120

x	1	2	3
y	1	0.5	0.33

$$\begin{aligned} \int_1^3 \frac{1}{x} \cdot dx &= \frac{1}{3} \left[(1+0.33) + 2(0.5) \right] \\ &= \frac{3.33}{3} = 1.111 \end{aligned}$$

Solutions of ordinary differential equations:

$$\frac{dy}{dx} = f(x, y), \quad x=x_0, \quad y=y_0, \quad f(x_1) = ?$$

It can be solved, by,

- (i) Taylor series method
- (ii) Picard's method.
- (iii) Runge-kutta method.

Q. Match the following:

E- Newton-Raphson

F- Runge-Kutta

G- Simpson's rule.

H- Gauss eliminator

I- solⁿ of non-linear equⁿ

II- solⁿ of linear algebraic equⁿ
(simultaneous equⁿ)

III- solⁿ of ordinary diff equⁿ

IV- Numerical integration.

E-I

F-III

G-IV

H-II

Probability:

If an experiment is conducted under essentially given conditions upto n times, let m cases are favourable to an event E then the probability of E is defined by $P(E)$ and is defined as

$$P(E) = \frac{\text{no. of favourable cases to the event } (m)}{\text{Total no. of events } (n)}$$

$$P(E) = \frac{m}{n}$$

$$\therefore P(\bar{E}) = \frac{n-m}{n}$$

$$= \frac{n}{n} - \frac{m}{n}$$

$$= 1 - P(E)$$

$$P(E) + P(\bar{E}) = 1$$

Axioms of probability.

- (i) Positivity, $0 \leq P(E) \leq 1$
- (ii) Certainty, $P(S) = 1$
- (iii) Union, $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Note:

If E_1, E_2 are mutually exclusive or disjoint

$$E_1 \cap E_2 = \phi$$

$$P(\phi) = 0$$

Sample space:

The set of all possible outcomes in an experiment is said to be the sample space.

e.g. If two coins are tossed

$$S = \{HH, HT, TH, TT\}$$

If dice is thrown.

$$S = \{1, 2, 3, 4, 5, 6\}$$

Conditional probability:

Let S be the sample space and E_1, E_2 are any two events of S : then the conditional probability of E_1 after the occurrence of E_2 is denoted by $P\left(\frac{E_1}{E_2}\right)$ and is defined as

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}, \quad P(E_2) \neq 0$$

Similarly

$$P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)}, \quad P(E_1) \neq 0$$

e.g. If two coins are tossed

$$S = \{HH, HT, TH, TT\}$$

If getting Head on any of coin is success then

Random variable, $X = \{0, 1, 2\}$ - subset of sample space

Mean of random variable:

If $\sum x_i \cdot P(X=x_i)$ exist, it is called mean of the random variable and is denoted by μ . (Expectation)

$$\text{Variance of } a \text{ or } V(\sigma^2) = \sum x_i^2 \cdot P(X=x_i) - \mu^2$$

Standard deviation:

It is the square root of variance and is denoted by σ .

Binomial distribution:

If, n - no. of trials.

p - probability of success

q - probability of failure.

$$p + q = 1$$

$$P(X=k) = {}^nC_k \cdot q^{n-k} \cdot p^k$$

If $p=q$.

$$P(X=k) = {}^nC_k \cdot p^n$$

Mean = np

$$\text{variance} = n \cdot p \cdot q$$

$$\text{standard deviation} = \sqrt{npq}$$

Poisson's distribution:

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda} \text{ where } \lambda \text{ is parameter.}$$

$$\text{Mean} = \lambda$$

$$\text{Variance} = \lambda$$

$$\text{S.D.} = \sqrt{\lambda}$$

Q. What is chance of leap year having 53 sundays.

$$\text{leap years} = 366 \text{ days.}$$

$$52 \times 7 = 364$$

$$P(E) = \frac{2}{7}$$

Q. For non-leap year having 53 Mondays

$$365 \text{ days}$$

$$P(E) = \frac{1}{7}$$

Q. A card is drawn from deck of playing cards & gambler bets that it is, a spade or ace. What are odds against his winning this bet (odds - ratio)

$$\text{No. of favourable cards} = 13 + (4 - 1) = 16$$

$\uparrow$ spades $\nwarrow$ spade ace
 $\swarrow$ Ace

$$\text{prob. of winning} = \frac{16}{52}$$

$$\text{losing} = 1 - \frac{16}{52} = \frac{36}{52}$$

$$\begin{aligned} \text{odds against winning} &= \frac{36}{52} : \frac{16}{52} \\ &= \frac{9}{4} \end{aligned}$$

Q. A card is drawn from (two cards) deck of playing cards at random. What is probability of both cards being king if

(i) First card is not replaced

(ii) Replacement is allowed.

$$\text{probability of drawing kings} = \frac{4}{52}$$

(i) without replacement

$$= \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

(ii) with replacement

$$= \frac{4}{52} \times \frac{4}{52} = \frac{1}{169}$$

Q. A husband and wife appear in an interview for two vacancies in same post. The probability of husband selection is $\frac{1}{7}$ & that of wife is $\frac{1}{5}$. What is probability that both

(i) both of them are selected

(ii) only one of them is selected

(iii) none of them selected

$$P(A) = \frac{1}{7} \quad P(B) = \frac{1}{5}$$

(22)

$$(i) \quad P(A \cap B) = P(A) \cdot P(B) = \frac{1}{7} \times \frac{1}{5}$$

$$P(\text{both getting selected}) = \frac{1}{35}$$

$$(ii) \quad P(\text{only one get selected}) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

$$= \frac{1}{7} \left(1 - \frac{1}{5}\right) + \frac{1}{5} \left(1 - \frac{1}{7}\right)$$

$$= \frac{4}{35} + \frac{6}{35}$$

$$= \frac{10}{35}$$

$$(iii) \quad P(\text{none of them selected}) = P(\bar{A} \cap \bar{B}) \quad (\text{don't do } 1-p)$$

$$= \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{7}\right) \quad \text{cause it includes indiv selection}$$

$$= \frac{4}{5} \times \frac{6}{7}$$

$$= \frac{24}{35}$$

A bag contain 3 red, 6 black, 7 white balls. What is the probability that 2 balls drawn are white & black.

$$= \frac{{}^6C_1 \times {}^7C_1}{{}^{16}C_2}$$

$$= \frac{6 \times 7}{\frac{16 \times 15}{2 \times 1}}$$

$$= \frac{7}{20}$$

There are 5 duplicate & 10 original items in automobile shop. 3 items are brought at random by customers. Find $P(\text{no item is duplicate})$

$$\frac{{}^{10}C_3}{{}^{15}C_3} = \frac{10 \times 9 \times 8}{15 \times 14 \times 13} = \frac{24}{91}$$

- Q. In a certain college 25% of students failed in mathematics and 15% failed in chemistry & 10% failed in both. If a student is selected at random, if he failed in chemistry then what is probability that he failed in maths. (i) Vice-versa, (ii) Neither maths nor chemistry.

$$P(M) = 0.25$$

$$P(C) = 0.15$$

$$P(M \cap C) = 0.10$$

Conditional probability,

$$(i) \quad P(M/C) = \frac{P(M \cap C)}{P(C)} = \frac{0.10}{0.15} = \frac{2}{3}$$

$$(ii) \quad P(C/M) = \frac{P(M \cap C)}{P(M)} = \frac{0.10}{0.25} = \frac{2}{5}$$

$$\begin{aligned} (iii) \quad P(\bar{M} \cup \bar{C}) &= 1 - P(M \cup C) \\ &= 1 - [P(M) + P(C) - P(M \cap C)] \\ &= 1 - [0.25 + 0.15 - 0.10] \\ &= 0.7 \end{aligned}$$

- Q. A problem in structure is given to students A, B & C whose chances of solving are $\frac{1}{2}$, $\frac{1}{3}$ & $\frac{1}{4}$ respectively. What is the probability that problem will be solved.

$$P(\text{problem will be solved})$$

$$= 1 - P(\text{no one solves})$$

$$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$$

$$= 1 - P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$$

$$= 1 - \left[\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \right]$$

$$= 1 - \left(\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \right)$$

$$= \frac{3}{4}$$

Q. A single die is thrown twice. find the probability that there sum is neither 8 nor 9. (122)

$$n(S) = 36.$$

for sum = 8 $\rightarrow E_1 = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

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$$P(E_1) = \frac{5}{36}$$

for sum = 9 $\rightarrow E_2 = \{(3,6), (4,5), (5,4), (6,3)\}$

$$P(E_2) = \frac{4}{36}$$

These are mutually exclusive events.

$$P(\bar{E}_1 \cap \bar{E}_2) = 1 - P(E_1 \cup E_2)$$

$$= 1 - [P(E_1) + P(E_2)]$$

$$= 1 - \left[\frac{5}{36} + \frac{4}{36} \right]$$

$$= \frac{3}{4}$$

Q. 4 fair coins are tossed simultaneously, find probability that at least one head and one tail, turn up.

$P(\text{at least one head \& one tail})$

$$= P(X=1) + P(X=2) + P(X=3)$$

$$= ({}^4C_1 + {}^4C_2 + {}^4C_3) \cdot \left(\frac{1}{2}\right)^4$$

$$= \frac{4+6+4}{16}$$

$$= \frac{7}{8}$$

$$P(X=k) = {}^nC_k \cdot p^k$$

or

$$P(E) = 1 - [P(X=4) + P(X=0)]$$

$$= 1 - \left[\left(\frac{4}{16} + \frac{1}{16} \right) \left(\frac{1}{2} \right)^4 \right]$$

$$= 1 - \left(\frac{1+1}{16} \right) = \frac{7}{8}$$

5 fair coins are tossed simultaneously. Find probability that at least one head turn up.

$$P(E) = P(X =$$

$$P(X \geq 1) = 1 - P(X = 0)$$

$$= 1 - {}^5C_0 \cdot \left(\frac{1}{2}\right)^5 = 1 - \frac{1}{32}$$
$$= \frac{31}{32}$$

Out of 800 families with 4 children each, how many families would be expected to have 2 boys & 2 girls

(i) at least 1 boy

(ii) at most 2 girls.

Family cannot be discriminated - can't go for poisson's distribution. (go for binomial)

$$n = 4 \quad p = q = \frac{1}{2}$$

$$(i) P(X=2) = {}^4C_2 \cdot \left(\frac{1}{2}\right)^4 = 6 \times \frac{1}{16} \times 800 = 300$$

$$(ii) P(X \geq 1) = 1 - P(X = 0)$$

$$= 1 - {}^4C_0 \cdot \left(\frac{1}{2}\right)^4 = 1 - \frac{1}{16}$$

$$= \frac{15}{16} \times 800 = 750$$

$$(iii) P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= ({}^4C_0 + {}^4C_1 + {}^4C_2) \cdot \left(\frac{1}{2}\right)^4$$

$$= \frac{1+4+6}{16} = \frac{11}{16} \times 800 = 550$$

Q. Suppose that a book of 300 pages containing 30 printing mistakes. Assuming that these errors are randomly distributed throughout the book & X is the no. of errors per page has a poisson distribution. What is the probability that 20 pages selected at random will be free of errors. (12)

$$p = \frac{30}{300} = \frac{1}{10}$$

$$n = 20$$

$$\lambda = n \cdot p$$

$$= 20 \times \frac{1}{10} = 2$$

No error is there, $p(X=0)$

$$e^{-\lambda} = e^{-2} = 0.135$$

Q. A box contains 3 blue balls & 4 red balls. Another identical box contains 2 blue balls & 5 red balls. One ball is picked at random from one of the two boxes & it is red. Probability that the ball came from first box is

$$\begin{aligned} \text{For identical boxes } P(E_1) &= P(E_2) = \frac{1}{2} \quad \text{for 2 boxes} \\ &= \frac{1}{3} \quad \text{for 3 boxes} \end{aligned}$$

$$P(\text{drawing red ball from 1st box}) = \frac{4}{7}$$

$$P(\text{drawing red ball from 2nd box}) = \frac{5}{7}$$

$$\begin{aligned} P(\text{red ball from 1st box}) &= \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)} \\ &= \frac{\frac{1}{2} \left(\frac{4}{7} \right)}{\frac{1}{2} \left(\frac{4}{7} \right) + \frac{1}{2} \left(\frac{5}{7} \right)} = \frac{4}{9} \end{aligned}$$

