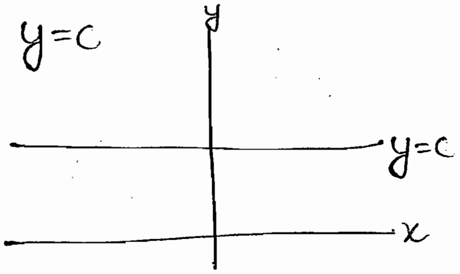
Topics : Weightage : 10-15 %.

1. Calculus.
2. Diff. Eq^{n}s.
3. Linear Algebra
4. Probability
5. Complex Analysis
6. Vector Calculus
7. Numerical Methods
8. Laplace Transforms

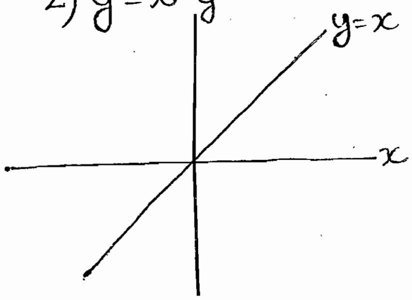
CALCULUS

Functions :

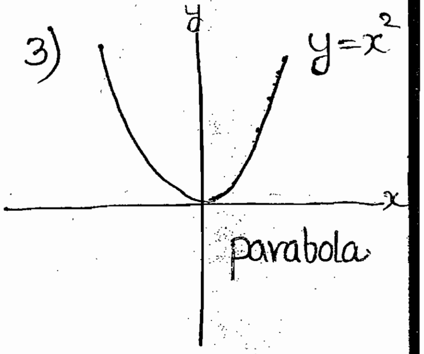
1) $y=c$



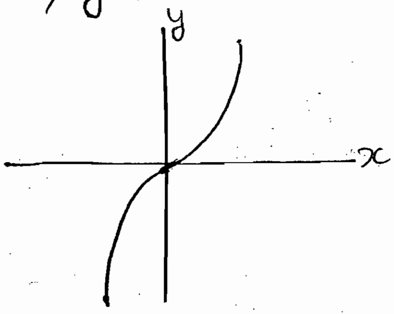
2) $y=x$



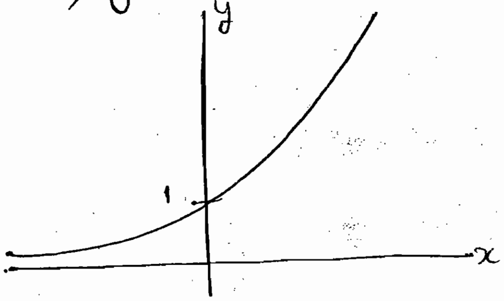
3) $y=x^2$



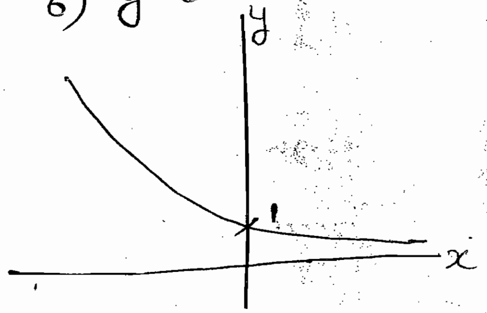
4) $y=x^3$



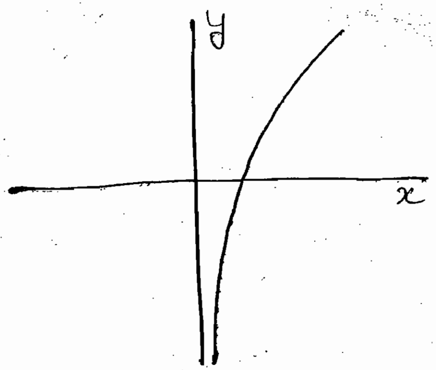
5) $y=e^x$



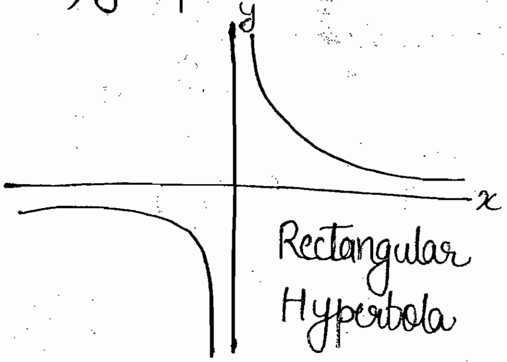
6) $y=e^{-x}$



7) $y=\log x$ (In Maths log is base e notation)

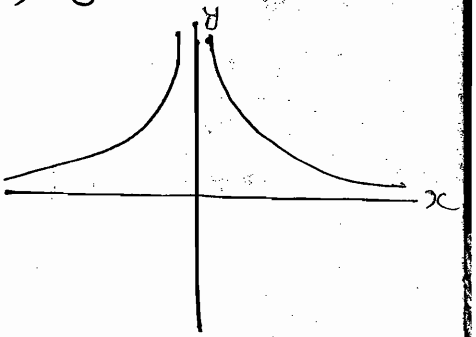


8) $y=1/x$



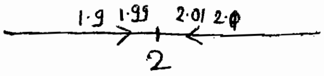
Rectangular Hyperbola

9) $y=1/x^2$



* $f(x) = \frac{x^2-4}{x-2}$ $\quad$ $f(2)$ is <u>not defined</u>.

$$\underset{x\to 2}{\text{Lt}}\ f(x) = \underset{x\to 2}{\text{Lt}}\ \frac{x^2-4}{x-2}$$



$x \to 2^-$		$x \to 2^+$	
$x = 1.9$	$f(x) = 3.9$	$f(x) = 4.1$	$x = 2.1$
$x = 1.99$	$f(x) = 3.99$	$f(x) = 4.01$	$x = 2.01$
$x = 1.999$	$f(x) = 3.999$	$f(x) = 4.001$	$x = 2.001$
	$\downarrow$ 4	$\downarrow$ 4	

$\underset{x\to a}{\text{Lt}}\ f(x) = L$ means $\underset{x\to a^-}{\text{Lt}}\ f(x) = \underset{x\to a^+}{\text{Lt}}\ f(x) = L$

i.e. $\boxed{L.H.L = R.H.L = L}$

Note – $f(a)$ need not be defined.

* $\underset{x\to 3}{\text{Lt}}\ 3x+4 = 13$ $\qquad$ $f(x) = 3x+4$
$f(3) = 13$

<u>Continuity</u> :–

<u>Def</u> : $f(x)$ is continuous at $x=a$ means

$\underset{x\to a}{\text{Lt}}\ f(x) = f(a)$ $\quad$ i.e. $\boxed{RHL = LHL = f(a)}$

otherwise, $f(x)$ is discontinuous at $x=a$

Removable discontinuity – $L.H.L = R.H.L \neq f(a)$

<u>Ex</u> :– $f(x) = \frac{x^3-9}{x-3}$ has removable discont. at $x=3$

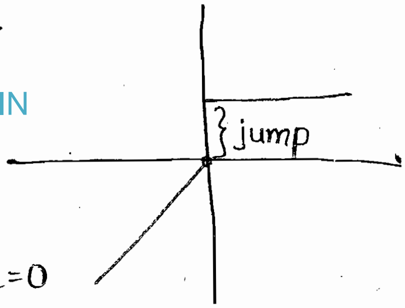
$\underset{x\to 3}{\text{Lt}}\ f(x) = 6 \neq f(3)$ $\qquad$ $f(3)$ not defined.

Jump discontinuity – (Discont. of first kind)

$$LHL \neq RHL$$

<u>Ex</u> :– $f(x) = \cancel{\frac{x}{x+1}} \begin{matrix} x & x<0 \\ x+1 & x\geq 0 \end{matrix}$

LHL $\neq$ RHL $\Rightarrow$ jump discontinuity at $x=0$.



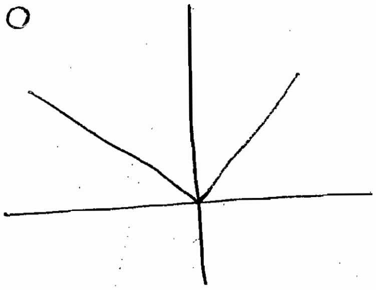
Ex:— $f(x) = |x| = \begin{cases} -x & x<0 \\ x & x \geq 0 \end{cases}$

a) Continuous only at $x=0$ c) Discont. only at $x=0$

[✓ b) Continuous $\forall x$] d) None

Sol^n:— At $x=0$; LHL $=0$; RHL $=0$; $f(0)=0$

LHL = RHL = $f(0)$



$|x|$ is continuous for $x<0$, $x>0$ & $x=0$

i.e. continuous for all x.

Results :—

1. Polynomial f^ns, exponential fns, $\sin x$, $\cos x$ are continuous for all x.

2. If f & g are continuous then
   (i) $f \pm g$ is continuous. (ii) fg is continuous
   (iii) f/g is continuous if $g \neq 0$.

Ex:— $f(x) = |x-1| + |x-2|$

$|x-1| = \begin{cases} -(x-1) & x<1 \\ x-1 & x \geq 1 \end{cases}$ $|x-2| = \begin{cases} -(x-2) & x<2 \\ x-2 & x \geq 2 \end{cases}$

$$f(x) = \begin{cases} -(x-1)-(x-2) & x<1 \\ (x-1)-(x-2) & 1 \leq x < 2 \\ (x-1)+(x-2) & x \geq 2 \end{cases}$$

$$f(x) = \begin{cases} -2x+3 & x<1 \\ 1 & 1 \leq x < 2 \\ 2x-3 & x \geq 2 \end{cases}$$

We check continuity at pt. $x=1$

$\because$ it is cont. at all other pts.

continuous at $x = 1$

At $x = 2$; LHL $= 1$; RHL $= 2(2) - 3 = 1$; $f(2) = 1$

Continuous at $x = 2$

$\Longrightarrow$ continuous $\forall x$.

Ex : $$f(x) = \begin{cases} \dfrac{1}{1+2^{1/x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Solⁿ :— At $x = 0$; $f(x) = 0$; LHL $= \underset{x \to 0^-}{\text{Lt}} \dfrac{1}{1+2^{1/x}} = \dfrac{1}{1+2^{-\infty}}$

RHL $= \underset{x \to 0^+}{\text{Lt}} \dfrac{1}{1+2^{1/x}} = \dfrac{1}{1+2^{\infty}} = \dfrac{1}{\infty} = 0$; LHL $= \dfrac{1}{1+0} = 1$

↳ $\frac{1}{x} \to \infty$ at $x \to 0^+$

As $1/x = -\infty$ at $x = 0^-$

LHL $\neq$ RHL $\Rightarrow$ not cont. at $x = 0$.

Ex :— $\underset{x \to 0}{\text{Lt}} \dfrac{1}{x}$ a) ∞ b) $-\infty$ ✓c) Does not exist

Solⁿ :— LHL $= \underset{x \to 0^-}{\text{Lt}} \dfrac{1}{x} = -\infty$; RHL $= \underset{x \to 0^+}{\text{Lt}} \dfrac{1}{x} = +\infty$

LHL $\neq$ RHL $\Rightarrow$ does not exist.

Q:- $$f(x) = \begin{cases} 2x+1 & x \leq 1 \\ ax^2+b & 1 < x < 3 \\ 5x+2a & x \geq 3 \end{cases}$$ This fⁿ is cont. $\forall x$ find a, b.

Solⁿ :— At $x = 1$; LHL $= 2(1)+1 = 3$; RHL $= a(1)^2+b = a+b$

cont. at $x = 1$ $\Rightarrow$ $a+b = 3$ ——— ①

At $x = 3$; LHL $= a(3)^2 + b = 9a+b$; RHL $= 5(3) + 2a = 15 + 2a$

cont. at $x = 2$ $\Longrightarrow$ $9a+b = 15+2a$ or $7a+b = 15$ —— ②

On solving ; $a = 2$ & $b = 1$

Differentiability — The derivative of f at x is defined as

$$f'(x) = \underset{h \to 0}{\text{Lt}} \frac{f(x+a) - f(x)}{h}$$ provided limit exist.

$f'(a)$ exist means $\boxed{\text{LHD} = \text{RHD}}$

Left hand Derivative = RH Derivative

2. $(x^n)' = nx^{n-1}$

3. $\left(\frac{1}{x}\right)' = \frac{-1}{x^2}$

4. $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

5. $(e^{ax})' = ae^{ax}$

6. $(\log x)' = \frac{1}{x}$

7. $(\sin x)' = \cos x$

8. $(\cos x)' = -\sin x$

10. $(\cot x)' = -\operatorname{cosec}^2 x$

11. $(\sec x)' = \sec x \tan x$

12. $(\operatorname{cosec} x)' = -\operatorname{cosec} x \cot x$

13. $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$

14. $(\tan^{-1} x)' = \frac{1}{1+x^2}$

15. $(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

Ex :- $f(x) = |x|$

a) Differentiable for all x b) Diff. only at $x=0$

[✓ c) Not Diff. at $x=0$] d) None

Soln :- $|x| = \begin{cases} -x & x<0 \\ x & x \geq 0 \end{cases}$ $f'(x) = \begin{cases} -1 & x<0 \rightarrow \text{LHD} \\ 1 & x \geq 0 \rightarrow \text{RHD} \end{cases}$

At $x=0$; LHD $\neq$ RHD $\Rightarrow$ not differentiable at $x=0$.

$*$ $(|x|)' = \frac{|x|}{x}$ $x \neq 0$

Ex :- $f(x) = |x-1| + |x-2|$

$f(x) = \begin{cases} -2x+3 & x<1 \\ 1 & 1 \leq x < 2 \\ 2x-3 & x \geq 2 \end{cases}$ $f'(x) = \begin{cases} -2 & x<1 \\ 0 & 1<x<2 \\ 2 & x>2 \end{cases}$

At $x=1$; LHD $= -2$ & RHD $= 0$; LHD $\neq$ RHD

Not differentiable at $x=1$.

At $x=2$; LHD $= 0$, RHD $= 2$; LHD $\neq$ RHD

Not Diff. at $x=2$

Q:- $f(x) = \begin{cases} x^2+3x+a & x \leq 1 \\ bx+2 & x>1 \end{cases}$

is differentiable for all x. Find a, b.

$2(1)+3 = b \quad \Rightarrow \quad b = 5$ —— ①

$$\boxed{\text{Differentiable} \Rightarrow \text{Continuous}}$$

$f(x)$ continuous at $x=1$; LHL = RHL

$$(1)^2 + 3(1) + a = b(1) + 2$$
$$a+4 = b+2$$
$$a-b = -2 \text{ ——— } ②$$

From ① , $a = 3$ & $b=5$ Ans.

Q:- A real f^n $f(x) = \begin{cases} \alpha x^2 + \beta x & x<0 \\ \alpha x^3 + \beta x^2 + 5\sin x & x \geq 0 \end{cases}$

is twice differentiable then $\alpha =$ 5 & $\beta =$ 5

Sol^n:- Cont. at $x=0$; RHL = LHL

LHL = 0 + 0 = 0 ; RHL = 0 + 0 + 0 ; 0 = 0

$$f'(x) = \begin{cases} 2\alpha x + \beta & x<0 \longrightarrow \text{LHD} = \beta \\ 3x^2\alpha + 2\beta x + 5\cos x & x \geq 0 \longrightarrow \text{RHD} = 5 \end{cases} \Rightarrow \beta = 5$$

$$f''(x) = \begin{cases} 2\alpha & x<0 \longrightarrow \text{LHD} = 2\alpha \\ 6\alpha x + 2\beta - 5\sin x & x \geq 0 \longrightarrow \text{RHD} = 2\beta \end{cases} \Rightarrow \cancel{2}\alpha = \cancel{2}\beta, \ \alpha = 5$$

$\Rightarrow \alpha = \beta = 5$ Ans.

Indeterminate Form:- $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \cdot \infty, 1^{\infty}, 0^0, \infty^0$

L'Hospital's rule : $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$

$\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Results — $\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$ 2. $\lim_{x \to 0} \frac{\tan x}{x} = 1$

3. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$ 4. $\lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}$ $(a>0)$

$x\to 0 \quad x \quad (0) \quad x\to 0$

② $$\underset{x\to 0}{Lt}\ \frac{\tan x}{x}\left(\frac{0}{0}\right) = \underset{x\to 0}{Lt}\ \frac{\sec^2 x}{1} = 1$$

③ $$\underset{x\to 0}{Lt}\ \frac{e^x-1}{x}\left(\frac{0}{0}\right) = \underset{x\to 0}{Lt}\ \frac{e^x}{1} = 1$$

④ $$\underset{x\to a}{Lt}\ \frac{x^n-a^n}{x-a}\left(\frac{0}{0}\right) = \underset{x\to a}{Lt}\ \frac{nx^{n-1}}{\cancel{1}} = na^{n-1}$$

Examples :—

⑤ $$\underset{x\to 0}{Lt}\ \frac{\tan 3x}{\sin 5x} = \underset{x\to 0}{Lt}\ \frac{\tan 3x/3x}{\sin 5x/5x}\times\frac{3}{5} = \frac{3}{5}\underset{x\to 0}{Lt}\ \frac{\tan 3x/3x}{\sin 5x/5x} = \frac{3}{5}$$

(Or) $$\underset{x\to 0}{Lt}\ \frac{3\sec^2 3x}{5\cos 5x} = \frac{3}{5}$$

Q:— $$\underset{x\to 0}{Lt}\ \frac{\sinh x-\sin x}{x\cdot\sin^2 x}$$

Sol^n:— $$\underset{x\to 0}{Lt}\ \frac{\sinh x-\sin x}{x\cdot x^2}\times\frac{1}{\frac{\sin^2 x}{x^2}}$$

$$\underset{x\to 0}{Lt}\ \frac{\sinh x-\sin x}{x^3}\left(\frac{0}{0}\right)\times\frac{1}{\boxed{\underset{x\to 0}{Lt}\ \frac{\sin^2 x}{x^2}}} = 1$$

L'rule :

$$\underset{x\to 0}{Lt}\ \frac{\cosh x-\cos x}{3x^2}\left(\frac{0}{0}\right)$$

L'rule :

$$\underset{x\to 0}{Lt}\ \frac{\sinh x+\sin x}{6x}\left(\frac{0}{0}\right)$$

L'rule :

$$\underset{x\to 0}{Lt}\ \frac{\cosh x+\cos x}{6}$$

$$= \frac{1+1}{6} = \frac{1}{3}$$ Ans.

Q:— $$\underset{x\to 0}{Lt}\ \frac{2^{8\cos x}\left[\sin^8\left(\frac{\pi}{6}+x\right)-\sin^8 \pi/6\right]}{8x}$$

$$\underset{x\to 0}{Lt}\ \frac{2^{8\cos x}}{8}\ \underset{x\to 0}{Lt}\ \frac{\sin^8\left(\frac{\pi}{6}+x\right)-\sin^8\frac{\pi}{6}}{x}\left(\frac{0}{0}\right.$$

$$= \frac{2^8}{8}\ \underset{x\to 0}{Lt}\ \frac{8\sin^7\left(\frac{\pi}{6}+x\right)\cos\left(\frac{\pi}{6}+x\right)-}{1}$$

$$= 2^8\sin^7\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)$$

$$= 2^8\left(\frac{1}{2}\right)^7\frac{\sqrt{3}}{2}$$

$$= \sqrt{3}$$ Ans.

x^3 finite, find a,b.

Sol^n :— $\underset{x\to 0}{Lt}\ \dfrac{2\cos 2x + a\cos x}{3x^2} = b$ $\left(\frac{0}{0}\right)$ $\Rightarrow \dfrac{2+a}{0} = b$ [finite]

Case (i) : $2+a \neq 0 \rightarrow b = \infty$ ×

Case (ii) : $2+a = 0 \Rightarrow a = -2$ ✓

$$\underset{x\to 0}{Lt}\ \frac{2\cos 2x - 2\cos x}{3x^2}\left(\frac{0}{0}\right) = \underset{x\to 0}{Lt}\ \frac{-4\sin 2x + 2\sin x}{6x}\left(\frac{0}{0}\right)$$

$$= \underset{x\to 0}{Lt}\ \frac{-8\cos 2x + 2\cos x}{6} = \frac{-8+2}{6} = -1 = b$$

$\therefore$ $a = -2$ & $b = -1$ Ans.

$1^\infty, 0^0, \infty^0$:— No direct form to solve these forms. So we take log.

$$y = f(x)^{g(x)} \Rightarrow \log y = g(x)\log f(x)$$

Results :—

1. $\underset{x\to 0}{Lt}\ (1+x)^{1/x} = e$
2. $\underset{x\to 0}{Lt}\ (1+ax)^{1/x} = e^a$
3. $\underset{x\to \infty}{Lt}\ \left(1+\frac{1}{x}\right)^x = e$
4. $\underset{x\to \infty}{Lt}\ \left(1+\frac{a}{x}\right)^x = e^a$
5. $\underset{x\to \infty}{Lt}\ (x)^{1/x} = 1$
6. $\underset{x\to 0^+}{Lt}\ x^x = 1$

① $\underset{x\to 0}{Lt}\ (1+x)^{1/x}$ (1^∞) ; $\log y = \underset{x\to 0}{Lt}\ \frac{1}{x}\log(1+x) = \underset{x\to 0}{Lt}\ \frac{\log(1+x)}{x}$ $(0/0)$

$\underset{x\to 0}{Lt}\ \dfrac{1/1+x}{1} \Rightarrow \log y = 1 \Rightarrow y = e$

② $\log y = \underset{x\to 0}{Lt}\ \frac{1}{x}\log(1+ax) = \underset{x\to 0}{Lt}\ \frac{\log(1+ax)}{x}\left(\frac{0}{0}\right)$

$\underset{x\to 0}{Lt}\ \dfrac{\frac{1}{1+ax}\cdot a}{1} \Rightarrow \log y = a \Rightarrow y = e^a$

$x \to \infty \quad x) \quad t$

$$\underset{t \to 0}{Lt} (1+t)^{1/t} = e$$

⑤ $y = \underset{x\to\infty}{Lt} (x)^{1/x} \quad (\infty)^0 \implies \log y = \underset{x\to 0}{Lt} \frac{\log x}{x} \left(\frac{\infty}{\infty}\right)$

$\log y = \underset{x\to 0}{Lt} \frac{1/x}{1} = 0 \quad \Rightarrow \quad \log y = 0 \Rightarrow y = e^{\circ} = 1$

⑥ $y = \underset{x\to 0}{Lt} \underset{(0^{\circ})}{x^x}$; $\log y = \underset{x\to 0}{Lt} \underset{[0\cdot\infty]}{x \log x} = \underset{x\to 0}{Lt} \frac{\log x}{1/x} \left(\frac{\infty}{\infty}\right)$

no std form

$\underset{x\to 0}{Lt} \frac{1/x}{-1/x^2} = 0 \Rightarrow y = e^{\circ} = 1$ 1st convert in std form

Ex:— $\underset{x\to\infty}{Lt} \left(\frac{1+x}{1+2x}\right)^x$

$$\underset{x\to\infty}{Lt} \left(\frac{x(1+1/x)}{x(1+2/x)}\right)^x = \frac{\underset{x\to\infty}{Lt} \left(1+\frac{1}{x}\right)^x}{\underset{x\to\infty}{Lt} \left(1+\frac{2}{x}\right)^x} = \frac{e}{e^2} = \frac{1}{e} \text{ Ans.}$$

Ex:— $\underset{x\to 0}{Lt} e^x (\cos x)^{1/\sin^2 x}$

$y = \underset{x\to 0}{Lt} e^x (\cos x)^{1/\sin^2 x} \implies \log y = \underset{x\to 0}{Lt} \left[\log e^x + \log (\cos x)^{1/\sin^2 x}\right]$

$\log y = \underset{x\to 0}{Lt} [x] + \underset{x\to 0}{Lt} \frac{\log \cos x}{\sin^2 x} \left(\frac{0}{0}\right) = 0 + \underset{x\to 0}{Lt} \frac{\frac{1}{\cos x} \cdot [-(\cancel{\sin x})]}{2\cancel{\sin x} \cos x}$

$\log y = \underset{x\to 0}{Lt} \frac{-1}{2\cos^2 x} = \frac{-1}{2} \Rightarrow y = e^{-1/2}$ Ans.

Ex:— $\underset{x\to\infty}{Lt} \left(\frac{5-x}{3-x}\right)^x = \underset{x\to\infty}{Lt} \left(\frac{1-5/x}{1-3/x}\right)^x = \frac{e^{-5}}{e^{-3}} = \frac{e^3}{e^5} = \frac{1}{e^2}$ Ans.

* $\underline{\infty - \infty}$ & $\underline{0 \cdot \infty}$ forms :

Ex :— $\underset{x\to\infty}{Lt} (e^{1/5x} - 1) \cdot x \; [0\cdot\infty] = \underset{x\to\infty}{Lt} \frac{e^{1/5x} - 1}{1/x} \left[\frac{0}{0}\right] = \underset{x\to\infty}{Lt} \frac{e^{1/5x} - 1}{5 \cdot \frac{1}{5x}}$

Let $\frac{1}{5x} = t \quad x \to \infty \Rightarrow t \to 0$

$\frac{1}{5} \underset{t\to 0}{Lt} \frac{e^t - 1}{t} = \frac{1}{5} \times 1 = \frac{1}{5}$ Ans.

$\to \frac{\pi}{2}$

$$= \underset{x\to\pi/2}{Lt}\left[\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right] = \underset{x\to\pi/2}{Lt}\left[\frac{1-\sin x}{\cos x}\right]\left[\frac{0}{0}\right]$$

<u>L'rule</u> : $\Rightarrow \underset{x\to\pi/2}{Lt} \frac{\cancel{-}\cos x}{\cancel{-}\sin x} = \frac{0}{1} = 0$ <u>Ans</u>.

Mean Value Theorems :— (MVT)

Rolle's MVT :— If $f(x)$ is defined in $[a,b]$ such that

(i) f is continuous in $[a,b]$

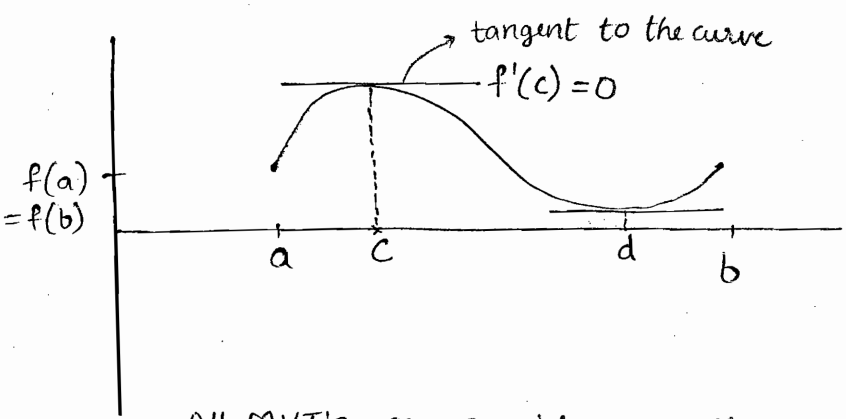
(ii) f is differentiable in (a,b)

(iii) $f(a) = f(b)$

$[a,b] : a \le x \le b \quad x \in R$

$(a,b) : a < x < b \quad x \in R$

then there exists alleast one $c \in (a,b)$ such that $\boxed{f'(c) = 0}$



All MVT's are special case of this MVT

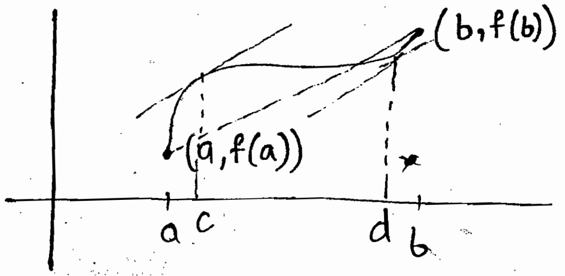
Lagranges MVT (1^{st} MVT) — If $f(x)$ is defined in $[a,b]$ such that

i) f is continuous in $[a,b]$.

ii) f is differentiable in (a,b).

then there exists atleast one $c \in (a,b)$ such that

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$



<u>Note</u> :— If $f(a) = f(b)$
Lagranges MVT reduces to
Rolle's MVT.

such that

(i) f & g are continuous in $[a,b]$

(ii) f & g are differentiable in (a,b)

(iii) $g' \neq 0$ in (a,b)

then there exists $c \in (a,b)$ such that $\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$

Q:-1. Find c of Rolle's MVT : $f(x) = x(x-1)(x-2)$ in $[1,2]$

2. $f(x) = \frac{\sin x}{e^x}$ in $[0,\pi]$ 3. $f(x) = |x|$ in $[-1,1]$

Solⁿ:- 1. $f(x) = x^3 - 3x^2 + 2x$ in $[1,2]$

(i) ∵ polynomial are cont. ∀x ; it is cont in $[1,2]$.

(ii) $f(x)$ is diff. in $(1,2)$ [∵ $f(x)$ is polynomial]

(iii) $f(1) = 0$; $f(2) = 0$

Three conditions of Rolle's MVT satisfied.

$f'(x) = 3x^2 - 6x + 2$

Find c such that $f'(c) = 0 \Rightarrow 3c^2 - 6c + 2 = 0$

$$c = \frac{6 \pm \sqrt{36-24}}{6} = \frac{6 \pm \sqrt{12}}{6}$$

$$= 1 \pm \frac{\sqrt{12}}{6} = 1 \pm \frac{\sqrt{2}}{\sqrt{6}} = 1 \pm \frac{1}{\sqrt{3}}$$

∵ $c \in (1,2)$

$\Rightarrow c = 1 + \frac{1}{\sqrt{3}}$ Ans.

2. $f(x) = \frac{\sin x}{e^x} = e^{-x}\sin x$

(i) f is continuous ∀x or in $[0,\pi]$

(ii) $f'(x) = e^{-x}\cos x - e^{-x}\sin x$ exists for all x or in $(0,\pi)$

(iii) $f(0) = 0$, $f(\pi) = 0$

Rolle's MVT is satisfied & applicable

$\sin n\pi = 0$
$\cos n\pi = (-1)^n$

$f'(c) = 0$; $e^{-c}\cos c - e^{-c}\sin c = 0$

$e^{-c}[\cos c - \sin c] = 0$

$e^{-c} \neq 0 \Longrightarrow \cos c = \sin c \Longrightarrow \tan c = 1$

$c = \frac{\pi}{4} \in (0,\pi)$ Ans.

$|x|$ is not diff. at $x=0$

$\therefore$ Rolle's MVT is not applicable.

Q:- Find c of Lagrange's MVT :

1) $f(x) = \log x$ in $[1, e]$ 3) $f(x) = \sqrt{x^2-4}$ in $[2,4]$

2) $f(x) = lx^2 + mx + n$ in $[a, b]$

<u>Sol</u> Solⁿ:- 1) $f(x) = \log x$ in $[1, e]$

i) $f(x)$ is cont. in $[1, e]$

ii) $f'(x) = \frac{1}{x}$ exists $(1, e)$ } Lagrange's MVT applicable.

Find c $\longrightarrow$ $f'(c) = \frac{f(e) - f(1)}{e-1} \Rightarrow \frac{1}{c} = \frac{1-0}{e-1}$

$\boxed{c = e-1} \in (1, e)$ Ans.

2) $f(x)$ is cont. in $[a, b]$ $\because$ polynomial f^n.

$f(x)$ is diff. in (a, b) $\because$ f is polynomial f^n

$f'(x) = 2lx + m$ exists in (a, b)

Lagrange's MVT is applicable.

Find c $\rightarrow$ $f'(c) = \frac{f(b) - f(a)}{b-a} \Rightarrow 2lc + m = \frac{lb^2 + mb + \cancel{n} - la^2 - ma - \cancel{n}}{b-a}$

$2\cancel{l}c + \cancel{m} = \frac{l[b^2 - a^2] + m[b-a]}{b-a} = \cancel{l}[b+a] + \cancel{m}$

$\boxed{c = \frac{a+b}{2}} \in (a, b)$ Ans.

★ 3) $f(x) = \sqrt{x^2-4}$ in $[2,4]$ → its continuity starts from 2 afterwards.

i) f is cont in $[2, 4]$

ii) $f'(x) = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-4}} = \frac{x}{\sqrt{x^2-4}}$ exists in $(2, 4)$ } Lagrange's MVT applicable.

$4-2$ $\sqrt{c^2-4}$ 2

$4c^2 = 12c^2 - 48 \Rightarrow 8c^2 = 48 \Rightarrow c^2 = 6 \Rightarrow c = \pm\sqrt{6}$

$c = \sqrt{6} \in (2,4)$ Ans.

Q:— Find c of Cauchy MVT :—

1) $f(x) = x^2$ $g(x) = x^3$ in $[1,2]$

2) $f(x) = e^x$ $g(x) = e^{-x}$ in $[a,b]$

Sol^n:— 1.(i) f, g cont. in $[1,2]$

(ii) $f' = 2x$, $g' = 3x^2$ exists in $(1,2)$

(iii) $g' = 3x^2 \neq 0$ in $(1,2)$ or $3x^2 = 0$ for $x = 0 \notin (1,2)$

Cauchy MVT is applicable.

Find c $\Rightarrow \dfrac{f'(c)}{g'(c)} = \dfrac{f(2)-f(1)}{g(2)-g(1)} \Rightarrow \dfrac{2c}{3c^2} = \dfrac{4-1}{8-1} \Rightarrow \dfrac{2}{3c} = \dfrac{3}{7}$

$c = \dfrac{14}{9} \in (1,2)$ Ans.

2. (i) f, g is cont. in $[a,b]$ or $\forall x$

(ii) $f' = e^x$, $g' = -e^{-x}$ exists (a,b) or $\forall x$

(iii) $g' = -e^{-x} = 0$ for $x \to \infty \notin (a,b)$

Cauchy MVT is applicable.

c: $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)} \Rightarrow \dfrac{e^c}{-e^{-c}} = \dfrac{e^b - e^a}{e^{-b} - e^{-a}}$

$\cancel{-}e^{2c} = \dfrac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = \dfrac{\cancel{e^b - e^a}}{\cancel{-}\,\cancel{e^a - e^b}} \times e^b e^a$

$c = \dfrac{a+b}{2} \in (a,b)$ Ans.

$\frac{a+b}{2} = c$

Taylor's Series :—

If f(x) is continuously differentiable at $x = a$ then f(x) can be expressed as power series in powers of $(x-a)$ [about $x = a$]

$$\boxed{f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots\cdots} \rightarrow$$

It is called Taylor's series in powers of $(x-a)$ [abt $x = a$ or with centre at $x = a$]

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \text{------}}$$

Ex :— $f(x) = \sin x$

$f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f''''(x) = \sin x$

$f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, $f''''(0) = 0$, $f^{v}(0) = 1$

$$\sin x = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \text{-----}$$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \text{------}}$$

Some power series :—

1) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{----}$

2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \text{------}$

3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{----}$

4) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{----}$

5) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \text{-----}$

6) $\sin hx = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \text{------}$

7) $\cos hx = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \text{--------}$

Ex :- $\sum_{n=0}^{\infty} \frac{1}{n!} =$

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \text{----} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Put $x = 1$; $\sum_{n=0}^{\infty} \frac{1}{n!} = e^1 = e$ Ans.

Sol^n :- $3\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}------\right]+2\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}------\right]$

$= 3+2x-3\frac{x^2}{2!}-2\frac{x^3}{3!}+3\frac{x^4}{4!}+2\frac{x^5}{5!}+------$ Ans.

Q:- For $x \ll 1$, cot h(x) can be expressed as

Sol^n :- $\cot h(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{\left[1+\frac{x^2}{2}+\frac{x^4}{4!}+----\right]}{\left[x+\frac{x^3}{3!}+\frac{x^5}{5!}+---\right]}$

For $x \ll 1$;
x^2, x^3, x^4 very small &
can be ignored. $\Rightarrow$ $\coth x = \frac{1}{x}$ Ans.

Q:- $\log\left(\frac{1+x}{1-x}\right) = \log(1+x)-\log(1-x)$

$= \left[x-\cancel{\frac{x^2}{2}}+\frac{x^3}{3}-\cancel{\frac{x^4}{4}}+----\right]-\left[-x-\cancel{\frac{x^2}{2}}-\frac{x^3}{3}-\cancel{\frac{x^4}{4}}----\right]$

$= 2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+----\right]$ Ans.

Q:- What is the coefficient $(x-2)^3$ in Taylor's series representation of e^x in powers of $(x-2)$

Sol^n :- $x-2=t \Rightarrow x=t+2$

$e^x = e^{t+2} = e^2 \cdot e^t = e^2\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+---\right]$

$= e^2\left[1+(x-2)+\frac{(x-2)^2}{2!}+\frac{(x-2)^3}{3!}+---\right]$

Coeff of $(x-2)^3 = \frac{e^2}{3!} = \frac{e^2}{6}$ Ans.

Q:- Expansion of $\frac{\sin x}{x-\pi}$ in powers of $(x-\pi)$.

Sol^n :- Put $x-\pi = t \Rightarrow x = t+\pi$

$\frac{\sin x}{x-\pi} = \frac{\sin(t+\pi)}{t} = \frac{-\sin t}{t} = \frac{-1}{t}\left[t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+-----\right]$

$= -1+\frac{t^2}{3!}-\frac{t^4}{5!}+\frac{t^6}{7!}------$

$x-\pi$ ... $\frac{}{3!} - \frac{(x-\pi)}{5!} + \frac{(x-\pi)}{7!} - - - -$ Ans.

Remember –

Silver (sin, cosec)	All
Tea (tan, cot)	Cups (cos, sec)

$$fn\left(n\cdot\frac{\pi}{2}\pm\theta\right) = \begin{cases} \pm f^n(\theta) & n \text{ even} \\ \pm \text{co-}f^n(\theta) & n \text{ odd} \end{cases}$$

f^n	co-f^n
sin	cos
tan	cot
sec	cosec

ex :– $\sin(\pi+\theta) = -\sin\theta$

$\cos(270^\circ-\theta) = -\sin\theta$

Partial Derivatives :—

Let $f(x,y) \longrightarrow f^n$ of two variables.

Notation : $\frac{\partial f}{\partial x} = f_x = p$; $\frac{\partial f}{\partial y} = f_y = q$; $\frac{\partial^2 f}{\partial x^2} = f_{xx} = r$

$\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = s$; $\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = s$; $\boxed{f_{xy} = f_{yx}}$

$\frac{\partial^2 f}{\partial y^2} = f_{yy} = t$

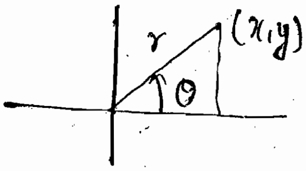
Ex :– $f(x,y) = x^3 + y^3 - 3axy$

$f_x = 3x^2 + 0 - 3ay$ $f_y = 0 + 3y^2 - 3ax$

$f_{xy} = f_{yx} = -3a$

$f_{xx} = 6x$, $f_{yy} = 6y$

Polar coordinates : $x = r\cos\theta$; $y = r\sin\theta$
(r,θ)



$\partial x^2 \qquad \partial y^2$

$$\theta = \tan^{-1}\frac{y}{x} \quad \& \quad r = \sqrt{x^2+y^2}$$

$$\theta_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{-y}{x^2}\right) = \frac{\cancel{x^2}\cdot(-y)}{x^2+y^2}\cdot\frac{1}{\cancel{x^2}} = \frac{-y}{x^2+y^2}$$

$$\theta_{xx} = \frac{-(-y)\cdot 2x}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\theta_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{x^{\cancel{2}}}{x^2+y^2}\cdot\frac{1}{\cancel{x}} = \frac{x}{x^2+y^2}$$

$$\theta_{yy} = \frac{-x\cdot 2y}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\Longrightarrow \boxed{\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0}$$

15 Dec 2014
L-2

Homogeneous function :-

$f(x,y) = x^3+y^3 \longrightarrow$ Homogeneous f^n of degree 3

$f(x,y) = x^3+x^2y+y^3 \longrightarrow$ H.f^n of degree 3

$f(x,y) = \dfrac{x^4+y^4}{x^2+y^2} \longrightarrow$ H.f^n of degree $4-2=2$

$f(x,y) = \dfrac{x^8+y^8}{\sqrt{x}+\sqrt{y}} \longrightarrow$ H.f^n of degree $8-\frac{1}{2}=\frac{15}{2}$

$f(x,y) = \dfrac{\sqrt{x}-\sqrt{y}}{x+y} \longrightarrow$ H.f^n of degree $\frac{1}{2}-1=\frac{-1}{2}$

ex :- $x^3+x^2y+1 \longrightarrow$ Not a H.f^n.

of degree n if

$$\left.\begin{array}{l}(1)\ f(Kx, Ky) = K^n f(x,y) \\ \qquad\qquad \text{or} \\ (2)\ f(x,y) = x^n \phi\left(\frac{y}{x}\right)\end{array}\right\} \ H\cdot f^n \text{ of degree } n.$$

ex :— $f(x,y) = x^3 + y^3$

$$f(kx, ky) = (kx)^3 + (ky)^3 = K^3\left[x^3+y^3\right] = K^{\textcircled{3}} f(x,y)$$

(or)

$$f(x,y) = x^3+y^3 = x^3\left[1+\left(\frac{y}{x}\right)^3\right] = x^{\textcircled{3}}\phi\left(\frac{y}{x}\right)$$

Euler's theorem :—

(I) If $f(x,y)$ is $H.f^n$ of degree n

$$\boxed{\begin{array}{ll}(1) & xf_x + yf_y = nf \\ (2) & x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1)f\end{array}}$$

Ex. $f(x,y) = \dfrac{x^3+y^3}{x+y}$

1) $xf_x + yf_y = 2.f(x,y)$

2) $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = 2(2-1)f = 2f$

(II) Let $u(x,y)$ is not $H.f^n$ but $f(u)$ is $H.f^n$ of degree "n".

$$\boxed{\begin{array}{ll}(1) & xu_x + yu_y = n\dfrac{f(u)}{f'(u)} = G(u) \\ (2) & x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = G(u)\left[G'(u) - 1\right]\end{array}}$$

Ex :— $u = \log\left(\dfrac{x^3+y^3}{x+y}\right)$ is not $H.f^n$ but $\dfrac{x^3+y^3}{x+y}$ is $H.f^n$

(1) $xu_x + yu_y = ?$

(2) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = ?$

$x+y$

$\therefore xu_x + yu_y = \frac{2e^u}{e^u} = 2$ Ans.

$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = G(u)\left[G'(u) - 1\right] = 2\left[0-1\right] = -2$ Ans.

(III) Let u is not H.fⁿ but it is sum of two fⁿ where

$U + V + W$

V is H.fⁿ of degree n & W is H.fⁿ of degree m.

$$\boxed{\begin{array}{ll} 1) & xu_x + yu_y = nV + mW \\ 2) & x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = n(n-1)V + m(m-1)W \end{array}}$$

ex :- $u = \frac{x^3+y^3}{x-y} + x\sin\left(\frac{y}{x}\right)$ 1) $xu_x + yu_y = 2\frac{x^3+y^3}{x-y} + x\sin\left(\frac{y}{x}\right)$

2) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 2\frac{x^3+y^3}{x-y} + 0$

V is H.fⁿ of degree 3-1 = 2, W is H.fⁿ of degree 1.

ex :- $\sin u = \frac{x+2y+3z}{x^8+y^8+z^8}$

$xu_x + yu_y + zu_z = ?$

sin u is H.fⁿ of degree 1-8=-7

$xu_x + yu_y + zu_z = n\frac{f(u)}{f'(u)}$

$= -7\frac{\sin u}{\cos u}$

$= -7\tan u$ Ans.

$u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$xu_x + yu_y = ?$

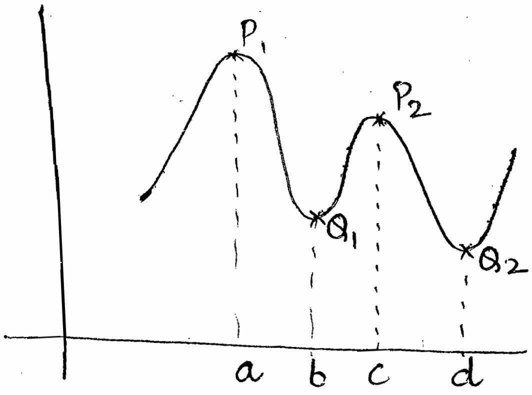
tan u is H.fⁿ of degree 3 - 1 = 2

$xu_x + yu_y = n\frac{f(u)}{f'(u)}$

$= 2\frac{\tan u}{\sec^2 u}$

$= 2\frac{\sin u \cos^2 u}{\cancel{\cos u}}$

$= \sin 2u$ Ans.



$y = f(x)$ is the f^n
Local max^m P_1 & P_2 at a & c respectively.

Local minimum Q_1 & Q_2 at b & d respectively.

→ At local maximum & minimum the tangent have zero slope & is parallel to x-axis. i.e. $\boxed{F'(x) = 0}$

→ To find maxima & minima : Idea → $\boxed{f'(x) = 0}$

Procedure :— Let $y = f(x)$ be given f^n.

1) Find $\frac{dy}{dx}$

2) Solve $\frac{dy}{dx} = 0$ for stationary pts.

3) Find $\frac{d^2y}{dx^2}$

4) Let "a" be stationary pt.
At $x = a$,

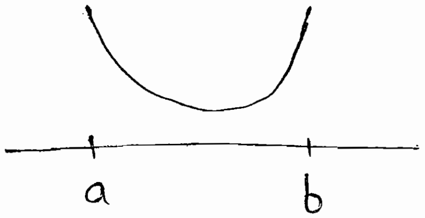
(i) If $\frac{d^2y}{dx^2} < 0$ then $f(x)$ has local max at $x = a$

(ii) If $\frac{d^2y}{dx^2} > 0$, then $f(x)$ has local min at $x = a$

(iii) If $\frac{d^2y}{dx^2} = 0$, test fails.

5) At $x = a$, If $\frac{d^2y}{dx^2} = 0$ & $\frac{d^3y}{dx^3} \neq 0$,
then $f(x)$ has point of inflection at $x = a$.

(1) If $\frac{d^2y}{dx^2} > 0$ in $[a, b]$ then $y = f(x)$ is concave up in $[a,b]$



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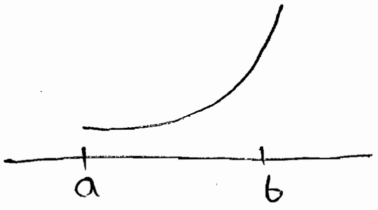
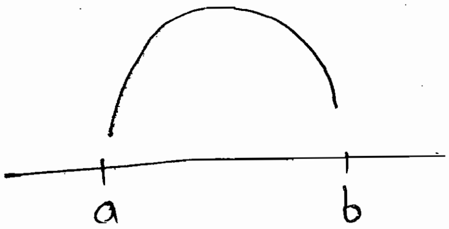


(2) If $\frac{d^2y}{dx^2} < 0$ in $[a, b]$ then $y = f(x)$ is concave down in $[a,b]$



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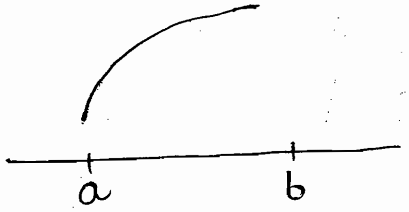
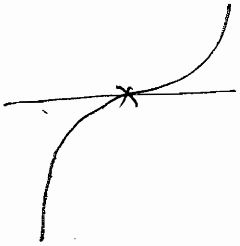


Def :- The pt. where a curve changes from concave up to concave down or vice versa is called pt. of inflection.



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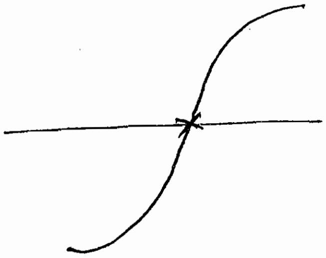


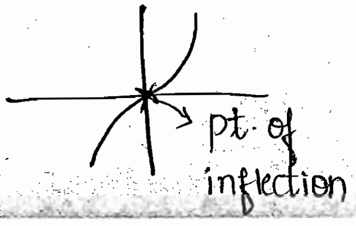
even at pt. of inflection tangent is parallel to x-axis

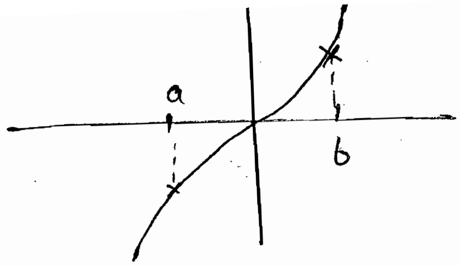
Ex :- $y = x^3$

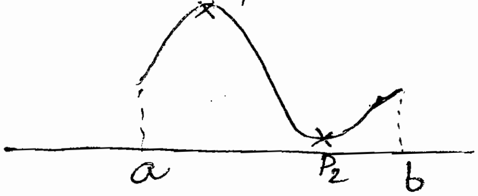
1) $\frac{dy}{dx} = 3x^2$ 2) $\frac{dy}{dx} = 0$ $3x^2 = 0$ $\Rightarrow$ $x = 0$ is stationary pt.

3) $\frac{d^2y}{dx^2} = 6x$ 4) at $x = 0$ $\frac{d^2y}{dx^2} = 0$ so we can't say whether it is maxima or minima.

5) $\frac{d^3y}{dx^3} = 6 \neq 0$ at $x = 0$ ∴ $y = x^3$ has pt. of inflection (neither maxima nor minima) at $x = 0$







Result :- 1) The absolute max^m of $f(x)$ in the $[a,b]$ is given by max^m $\{ f(a), f(b)$, all local max^m of $f(x)$ in $[a,b] \}$

2) The absolute min^m of $f(x)$ in ~~in~~ $[a,b]$ is given by Min^m $\{ f(a), f(b)$, all local min^m of $f(x)$ in $[a,b] \}$.

Ex :- Discuss max, min of $f(x) = x^3 - 9x^2 + 24x + 5$

Sol^n :- 1) find $f'(x)$: $f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8)$

2) $f'(x) = 0 \Rightarrow x^2 - 6x + 8 = 0 \Rightarrow x = 2, 4$ → stationary pts.

3) $f''(x) = 6x - 18$

4) $f''(x)\Big|_{x=2} = -6 < 0$ local max^m

$f''(4) = 6 > 0$ local min^m

Local max of $f(x) = f(2) = 2^3 - 9(2)^2 + 24(2) + 5 = 25$ at $x = 2$

Local min of $f(x) = f(4) = (4)^3 - 9(4)^2 + 24(4) + 5 = 21$ is at $x = 4$.

Q :- Find absolute max, min of $f(x) = x^3 - 9x^2 + 24x + 5$ in $[1,6]$

Sol^n :- 1) $f'(x) = 3x^2 - 18x + 24$

2) $f'(x) = 0 \Rightarrow x = 2, 4$ → stationary pts.

3) find $f(1)$, $f(2)$, $f(4)$, $f(6)$

$f(1) = 21$, $f(2) = 25$, $f(4) = 21$, $f(6) = 41$

Absolute Max^m = Max^m of $\{ f(1), f(6); f(2), f(4) \}$

$= f(6) = 41$ is at $x = 6$

Q:- Find max, min of $f(x) = \frac{\log x}{x}$ in $(0, \infty)$

<u>Sol^n</u>:- $f'(x) = \frac{x \frac{1}{x} - \log x}{x^2} = \frac{1 - \log x}{x^2}$

$$f'(x) = \frac{1 - \log x}{x^2} = 0 \implies 1 - \log x = 0 \implies \log x = 1 \implies \boxed{x = e}$$

$$f''(x) = \frac{x^2\left(\frac{-1}{x}\right) - 2x(\log x + 1)}{x^4} = \frac{\cancel{-x} - 2\cancel{x}(\log x+1)}{x^{\cancel{4}3}} = \cancel{-\frac{(1+2\log x)}{x^3}}$$

$$= \frac{-3x + 2x\log x}{x^4} = \frac{-3 + 2\log x}{x^3}$$

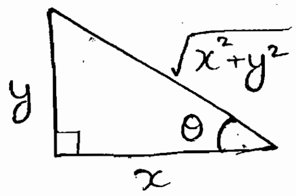
$$f''(e) = \frac{-3+2}{e^3} = \frac{-1}{e^3} < 0$$

$f(x)$ has local max^m at $x = e$ & local max^m of $f(x) = f(e)$

$= \frac{\log e}{e} = \frac{1}{e}$ <u>Ans</u>.

Q:- For right angle triangle if the sum of the lengths of the hypotenuse & a side kept const in order to have max^m area of the triangle; the angle b/w the hypotenuse & the side ———

<u>Sol^n</u>:-



$\sqrt{x^2+y^2} + x = \text{const.} = c$ ——— ①

Area = Max^m

Area of right angle Δ's $= \frac{1}{2}xy$

$A^2 = \frac{1}{4}x^2y^2$ ——— ②

① $\sqrt{x^2+y^2} = c - x$

squaring, $\cancel{x^2} + y^2 = c^2 - 2cx + \cancel{x^2} = c^2 - 2cx$

Let $f(x) = A^2 = \frac{1}{4}x^2y^2$

$f(x) = \frac{1}{2}x^2(c^2 - 2cx)$

$f(x) = \frac{1}{4}(c^2x^2 - 2cx^3)$

2) $f'(x) = 0 \Rightarrow x = 0, \frac{c}{3} \longrightarrow$ stationary pts.

3) $f''(x) = \frac{1}{4}[2c^2 - 12cx]$

4) $f''(0) = \frac{2c^2}{4} > 0$ local min^m

$f''\left(\frac{c}{3}\right) = \frac{-2c^2}{4} < 0$ local max^m ✓

$x = \frac{c}{3}$; $f(x)$ is max ; $f\left(\frac{c}{3}\right) = A^2 = \frac{1}{4}x^2y^2$

$y^2 = c^2 - 2cx = c^2 - 2c.\frac{c}{3} = \frac{c^2}{3} \Rightarrow y = \pm\frac{c}{\sqrt{3}}$

$y \rightarrow$ +ve side $\rightarrow + \frac{c}{\sqrt{3}}$

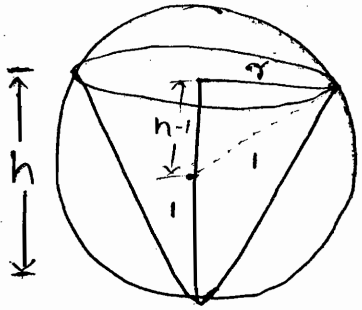
$\tan\theta = \frac{y}{x} = \frac{c/\sqrt{3}}{c/3} = \sqrt{3} \Rightarrow \theta = 60°$ or $\frac{\pi}{3}$

Q:— The right circular cone of largest volume that can be enclosed by a sphere of rad. 1m has a height of ——

a) $\frac{1}{3}$m b) $\frac{2}{3}$m c) $\frac{2\sqrt{2}}{3}$m [✓ d) $\frac{4}{3}$m.]

Sol^n:— Volume of cone,

$V = \frac{1}{3}\pi r^2 h$



By Right angle Δ,

$(h-1)^2 + r^2 = 1^2$

$r^2 = 1^2 - (h-1)^2$

$= \cancel{1^2} - h^2 - \cancel{1^2} + 2h$

$r^2 = 2h - h^2$

$V = \frac{1}{3}\pi(2h-h^2)h = \frac{\pi}{3}(2h^2 - h^3)$

$V'(h) = \frac{\pi}{3}[4h - 3h^2]$

$v''(h) = \frac{\pi}{3}[4-6h]$ at $h=0 \Rightarrow v''(0) = \frac{4\pi}{3} > 0$ local min^m

Ans $\boxed{\text{at } h=\frac{4}{3}}$; $v''\left(\frac{4}{3}\right) = \frac{-2\pi}{3} < 0$ local max^m

Q:— $f(x) = 2x^3 - 9x^2 + 12x - 3$ in $[0,3]$

Soln :— $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$

$f'(x) = 0 \Rightarrow x^2 - 3x + 2 = 0 \Rightarrow x^2 - 2x - x + 2 = 0$

$x(x-2) - 1(x-2) = 0$

$x = 2, 1 \rightarrow$ st. pts.

$f''(x) = 12x - 18$

$f''(1) = 12 - 18 = -6 < 0 \rightarrow$ local max^m

$f''(2) = 24 - 18 = 6 > 0 \rightarrow$ local min^m

$f(0) = -3$, $f(1) = 2 - 9 + 12 - 3 = 2$, $f(2) = 16 - 36 + 24 - 3 = 40 - 3 = 1$

$f(3) = 54 - 81 + 36 - 3 = 6$

Absolute max^m of $f(x) = 6$; Absolute min^m of $f(x) = -3$

Abs. max^m at $x = 3$; Abs. min^m is at $x = 0$

Function of two variables :—

To find local max, min,

1) Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$

2) Solve $f_x = 0$ & $f_y = 0$ for stationary pts. (Critical pts.)

3) Find $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$

4) Let (a,b) be stationary pt. & at (a,b)

(i) If $rt - s^2 > 0$ &

  (a) $r > 0$ then $f(x,y)$ has local min^m at (a,b)

  (b) $r < 0$ then $f(x,y)$ has local max^m at (a,b)

(ii) If $rt - s^2 = 0$, test fails.

(neither $\max^m$ nor minimum) at (a, b)

Ex :- $f(x,y) = 2x^4 + y^2 - x^2 - 2y$

Sol^n :- 1) Find f_x & f_y ;

$f_x = 8x^3 - 2x$, $f_y = 2y - 2$

2) $f_x = 0 \Rightarrow x = 0 , x = \pm\frac{1}{2}$; $f_y = 0 \Rightarrow y = 1$

St. pts $\rightarrow (0,1)$; $(+\frac{1}{2}, 1)$; $(\frac{-1}{2}, 1)$

3) $r = 24x^2 - 2$; $s = 0$; $t = 2$

4)

St. Pts	$r = 24x^2 - 2$	$t = 2$	$s = 0$	$rt - s^2$	so what = ?
$(0,1)$	$r = -2$	2	0	-4	Saddle
$(\frac{1}{2}, 1)$	$r = 4$	2	0	8	local $\min^m$
$(\frac{-1}{2}, 1)$	$r = 4$	2	0	8	

Q:- The distance b/w origin & the pt. nearest to it on the surface $z^2 = 1 + xy$ is ____

Sol^n:- Distance from origin to (x, y, z)

$$d = \sqrt{x^2 + y^2 + z^2}$$

$\therefore$ on the surface $z^2 = 1 + xy$

$$d^2 = x^2 + y^2 + 1 + xy = f(x,y) \text{ (Let)}$$

$f_x = 2x + y$, $f_y = 2y + x$ $\Rightarrow$ $2x + y = 0$ & $2y + x = 0$

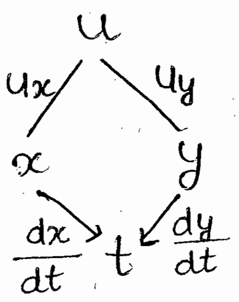
$r = 2$, $s = 1$, $t = 2$ $\Rightarrow \boxed{y = 0 , x = 0}$

$rt - s^2 = 4 - 1 = 3 > 2$ & $r > 0 \rightarrow$ local $\min^m$.

Local $\min^m$ distance $= d = \sqrt{0^2 + 0^2 + z^2}$

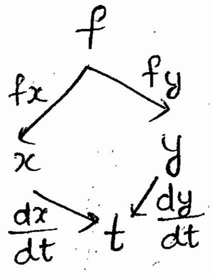
$d^2 = (1 + xy) \Rightarrow d = 1$ Ans.

Let $u = f(x,y)$ & $x = f_1(t)$ & $y = f_2(t)$



$$\boxed{\begin{aligned}\frac{du}{dt} &= \text{Total derivative of } u \text{ w.r.t } t\\ &= \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}\end{aligned}}$$

Ex :– $f(x,y) = 2xy$; $x = t$, $y = t^2$



$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= 2y\cdot 1 + 2x\,2t$$

$$= 2y + 4xt$$

$$= 2t^2 + 4t^2$$

$$= 6t^2$$ Ans.

Explicit Function – $y = f(x)$

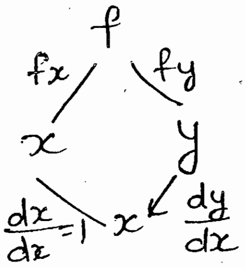
Implicit Function – $f(x,y) = c$

ex → $y^2 + y + \sin x = 0$ → Writing y in terms of x is complex ⟹ Implicit function.

Derivative of Implicit F^n :–

$$f(x,y) = c$$

To find $\frac{dy}{dx}$; Total derivative w.r.t. x



$$\frac{df}{dx} = 0 \quad ; \quad f_x\cdot 1 + f_y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

Ex :– $x^3 + y^3 + 3axy = 0$; $\frac{dy}{dx} = 0$ ⟹ $f = x^3 + y^3 + 3axy$

$$\frac{dy}{dx} = \frac{-(3x^2 + 3ay)}{3y^2 + 3ax} = \frac{-(x^2 + ay)}{y^2 + ax}$$ Ans.

Solⁿ:— $y = \sqrt{\tan x + y} \Rightarrow y^2 = \tan x + y \Rightarrow y^2 - y - \tan x = 0$

Even fⁿ given is in explicit form, it is a implicit fⁿ.

$$\frac{dy}{dx} = -\frac{(-\sec^2 x)}{2y-1} = \frac{\sec^2 x}{2y-1} \quad \text{Ans.}$$

Q:— $x^a y^b = (x+y)^{a+b}$ Find dy/dx

Solⁿ:— Take log on both sides →

$$\log(x^a) + \log(y^b) = (a+b)\log(x+y)$$

$$a\log x + b\log y - (a+b)\log(x+y) = 0$$

$$f_x = \frac{a}{x} - \frac{(a+b)}{x+y} = \frac{ay-bx}{x(x+y)}$$

$$f_y = \frac{b}{y} - \frac{(a+b)}{x+y} = \frac{bx-ay}{y(x+y)}$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(ay-bx)}{x(x+y)} \Big/ \frac{(bx-ay)}{y(x+y)} = \frac{y}{x} \quad \text{Ans.}$$

Total Differential — Let $u = f(x,y)$

The total Diff. of u :

$$\boxed{du = \frac{\partial u}{\partial x}\cdot dx + \frac{\partial u}{\partial y}dy}$$

ex:— 1) $d(xy) = \frac{\partial}{\partial x}(xy)dx + \frac{\partial}{\partial y}(xy)dy = ydx + xdy$

2) $d(x^2y) = y\cdot 2xdx + x^2dy = 2xydx + x^2dy$

3) $d(x^2y^2) = 2xy^2dx + 2yx^2dy$

4) $d\left(\frac{y}{x}\right) = \frac{x\cdot dy - ydx}{x^2}$

5) $d(\tan^{-1}(x^2+y^2)) = \frac{1}{1+(x^2+y^2)^2}\cdot d(x^2+y^2) = \frac{1}{1+(x^2+y^2)^2}[2xdx + 2ydy]$

x^4

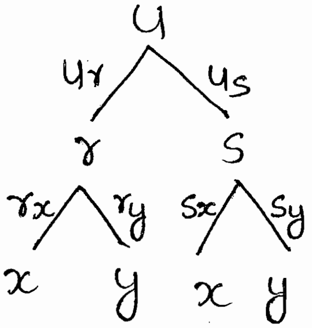
~~DIFFERENTIAL EQUATION~~

D.E :—

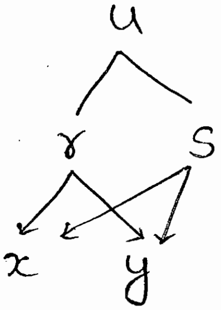
$y\,dx + x\,dy = 0 \Rightarrow d(xy) = 0 \Rightarrow \boxed{xy = c}$

Change of variable (chain rule) :—

Let $u = f(r,s)$ & $r = f_1(x,y)$; $s = f_2(x,y)$



(or)



$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= u_r \cdot r_x + u_s \cdot s_x \\ \frac{\partial u}{\partial y} &= u_r \cdot r_y + u_s \cdot s_y \end{aligned}}$$

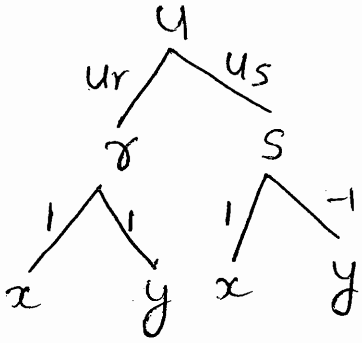
Ex :— $u = f(r,s)$; $r = x+y$ & $s = x-y$; $u_x + u_y =$ ______

Sol^n :— $u_x + u_y$

$u_x = u_r \cdot 1 + u_s \cdot 1 = u_r + u_s$

$u_y = u_r \cdot 1 + u_s(-1) = u_r - u_s$

$u_x + u_y = 2u_r$ Ans.



Integration —

1) $\int x^n dx = \frac{x^{n+1}}{n+1}$ 2) $\int \sin x\,dx = -\cos x$

3) $\int \tan x\,dx = \log \sec x$ 4) $\int \sec^2 x\,dx = \tan x$

5) $\int \sec x \tan x\,dx = \sec x$ 6) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}$

$\int \; a^2+x^2 \qquad a \qquad a \qquad \int \frac{}{f(x)}$

9. $\int \cot x\,dx = \log \sin x$ 10. $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$

Ex :- $\int \tan x\,dx = -\int \frac{-\sin x}{\cos x}dx = -\log(\cos x) = \log\sec x$

Ex :- $\int (\log x)^2 \cdot \frac{1}{x}dx = \frac{(\log x)^3}{3}$

11. $\int e^{f(x)} f'(x)\,dx = e^{f(x)}$ Ex :- $\int e^{x^2} x\,dx = \frac{1}{2}\int e^{x^2} 2x\,dx$

$= \frac{e^{x^2}}{2}$

(or) put $x^2 = t$ $2x\,dx = dt$

$= \frac{1}{2}\int e^t dt = \frac{1}{2}et = \frac{1}{2}e^{x^2}$

12. $\int e^x [f(x) + f'(x)]\,dx = e^x f(x)$

ex :- $\int e^x(\sin x + \cos x)\,dx = e^x \sin x$

Integration by parts :- ILATE → Exponential

I ↓ Inverse

L ↓ Log

A ↓ Algebraic

T ↓ Trignometric

$$\boxed{\int \overset{I}{u}\cdot\overset{II}{v} = u\int v - \int \cancel{dx}\,[u'\int v]}$$

Ex :- $\int \log x\,dx = \int \log x \cdot 1\,dx$

$= \log x\,[x] - \int \frac{1}{\cancel{x}}\cdot\cancel{x}\,dx = \log x\,[x] - x$

$= x(\log x - 1)$ Ans.

$$\boxed{\int \log x = x(\log x - 1)}$$

$x^{1/2+1}$

Q :- $\int \sqrt{x}\cdot\log x\,dx = \int \log x\cdot\sqrt{x}\,dx$

$= \log x\left[\frac{x^{3/2}}{3/2}\right] - \int \frac{1}{x}\cdot\frac{x^{3/2}}{3/2}\,dx$

Leibniz Rule — Useful when $u =$ polynomial f^n ~~or~~ i.e. $x^n; n=+ve$ (or) $v = e^{ax}$ or $\sin ax$ or $\cos ax$

$$\boxed{\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots}$$

where $v_1, v_2, v_3, v_4 \ldots$ are successive integrations & $u', u'' \ldots$ are successive differentiations.

ex :- $\int e^{2x}x^2\,dx = x^2\left[\frac{e^{2x}}{2}\right] - 2x\left[\frac{e^{2x}}{4}\right] + 2\left[\frac{e^{2x}}{8}\right] + 0$

ex :- $\int x^2 e^{-3x}\,dx = x^2\left[\frac{e^{-3x}}{-3}\right] - 2x\left[\frac{e^{-3x}}{9}\right] + 2\left[\frac{e^{-3x}}{-27}\right]$

Q :- $\int x^3 \sin x\,dx = x^3(-\cos x) - 3x^2(-\sin x) + 6x(\cos x) - 6\sin x$

Q :- $\int x^2 \cos 2x\,dx = x^2\left(\frac{\sin 2x}{2}\right) - 2x\left(\frac{-\cos 2x}{4}\right) + 2\left(-\frac{\sin 2x}{8}\right)$

DIFFERENTIAL EQUATIONS

D.E. is an eq^n involving derivatives of dependent variables w.r.t one or more independent variables.

DE

Ordinary DE (ODE or DE)
→ one independent variable

Ex :- $\frac{d^2y}{dx^2} + 9\frac{dy}{dx} + y = \sin x$

Partial DE (PDE)
→ Two or more independent variables.

Ex :- $\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial u}{\partial t}$

Order :— It is order of highest order derivative occuring in D.E.

Degree :— The power of highest order derivative when the dependent variables & its derivatives in the DE are free of fractional powers.

$$\left[\quad\left(\frac{}{dx}\right)\right] = \left(\frac{d\ y}{dx^2}\right)$$

Squaring both sides $\longrightarrow$ $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^{②}y}{dx^2}\right)^{②}$ → Order → Degree

Order = 2 & Degree = 2

Ex:— $\left(\frac{d^2y}{dx^2}\right)^3 + 3\frac{dy}{dx} + y = \sqrt{\sin x}$

Order = 2 , degree = 3

Formation of DE :— [By eliminating arbitrary constants (parameters)]

Method :—

1) $f(x, y, c_1, c_2) = 0$ ① Two parameter family of curves
Diff. twice & eliminate c_1 & c_2

2) Diff. w.r.t. x: $F'(x, y, c_1, c_2) = 0$ —— ②

3) Diff. w.r.t. x : $F''(x, y, c_1, c_2) = 0$ —— ③

4) Eliminate c_1 & c_2 using ①, ② & ③

$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \longrightarrow$ 2nd order D.E. ~~of~~ family.

Having n constants → n times differentiation → nth order DE

Q:- $y = mx$

Soln:— m → arbitrary constant

$\frac{dy}{dx} = m \implies y = \frac{dy}{dx}x$ or $\boxed{x\frac{dy}{dx} - y = 0}$

↓
D.E. of the family of st. lines.

Q:- $y = Ax + Bx^2$

$\left[\frac{dy}{dx} = A + 2Bx \ ; \ \frac{d^2y}{dx^2} = 2B\right]$ (or) $\frac{y}{x} = A + Bx$

Diff. w.r.t. x

$\frac{xy' - y.1}{x^2} = B$

Again Diff. w.r.t. x

$\frac{x^2[xy'' + y' - y'] - [xy' - y].2x}{x^4} = 0 \longrightarrow$ D.E. ↳ $x^2y'' - 2xy' + 2y = 0$

dx

$$\frac{d^2y}{dx^2} \longrightarrow y'' \longrightarrow D^2y \qquad D^2 = d^2/dx^2$$

$$\vdots$$

$$\frac{d^ny}{dx^n} \longrightarrow y^{(n)} \longrightarrow D^ny$$

Result :-

:- $y = C_1e^{ax} + C_2e^{bx}$ ⟶ C_1, C_2 : arbitrary constant.

D.E. ⇒ $y'' - (a+b)y' + (ab)y = 0$ ⟶ For only these type of prblms!

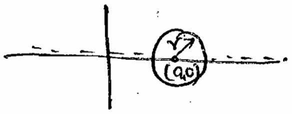
ex :- $y = C_1e^{2x} + C_2e^{3x}$

D.E. ⟹ $y'' - 5y' + 6y = 0$

ex :- $y = C_1e^{-x} + C_2e^{3x}$

D.E. ⇒ $y'' - 2y' - 3y = 0$

Ex :- Form DE of family of circles with radius r & having their centre on X-axis.



Solⁿ :- r → fixed → its a constant

a → varying → parameter or arbitrary constant.

Eqⁿ of circle with centre $(a,0)$ & radius r

$$(x-a)^2 + y^2 = r^2 \quad \text{———} \; ①$$

Diff. w.r.t. x ⟶ $2(x-a) + 2yy' = 0 \Rightarrow (x-a) = -yy'$

$$(-yy')^2 + y^2 = r^2 \quad \text{or} \quad y^2y'^2 + y^2 = r^2$$

$$\boxed{y^2(1+y'^2) = r^2}$$ Ans. $y' \rightarrow dy/dx$

Solutions of D.E. :-

Ex :- $\frac{dy}{dx} - 2y = 0$ ——— ①

Solⁿ :- $y = e^{2x}$ (Let) $\frac{dy}{dx} = 2e^{2x}$ Put in ① ⟹ $2e^{2x} - 2e^{2x} = 0$

D.E. is satisfied

$y = e^{2x}$ is the solⁿ.

satisfying D.E.

Solution is of two type — General sol^n & Particular Sol^n.

General sol^n – A sol^n of n^{th} order D.E. ~~eliminating~~ containing n arbitrary constants (parameters) (or)

A sol^n of DE containing

$\boxed{\text{No. of arbitrary constants = Order of DE}}$

Particular Sol^n – A sol^n of DE obtained from general solution

Note – To get particular sol^n of $\underline{n^{th}\text{ order}}$ D.E. we need $\underline{n}$ <u>distinct</u> conditions.

Initial conditions – Conditions given at same pt.

Boundry conditions – Conditions given at different pt.

Initial Value Prblms (IVP) : DE + Initial Conditions

Boundry Value Prblms (BVP) : DE + Boundry Conditions

Solutions of 1st order DE :—

<u>Methods</u> :— 1) Variable Separable 2) Exact DE

3) Linear DE of 1st order.

Variable Separable :—

<u>Ex</u> :— $\frac{dy}{dx} = \frac{x}{y} \Rightarrow ydy = xdx$ Integrating both sides,

$$\int y\,dy = \int x\,dx + c$$

$$\boxed{\frac{y^2}{2} = \frac{x^2}{2} + c} \rightarrow \text{gen. Sol}^n \quad \text{or} \quad \boxed{y^2 - x^2 = c} \rightarrow \text{general sol}^n$$

$dx \quad y$

Gen. Sol^n

$$\frac{y^2}{2} = \frac{x^2}{2} \mp c$$

$y(0) = 1 \rightarrow x=0$, $y=1$

$c = 1/2$

$$\frac{y^2}{2} = \frac{x^2}{2} + \frac{1}{2}$$

$$\boxed{y^2 - x^2 = 1}$$

Gen. Sol^n

$y^2 - x^2 = c$

$y(0) = 1 \quad x=0, y=1$

$1 - 0 = c$

$c = 1$

$$\boxed{y^2 - x^2 = 1}$$

same sol^n

<u>Ex</u> :- 1. $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + \cos y}$ 2. $e^x \tan y\, dx + (1+e^x)\sec^2 y\, dy = 0$

3. $\frac{dy}{dx} = 1 + x + y + xy$

<u>Sol^n</u> :- 1. $(\sin y + \cos y)\,dy = x(2\log x + 1)\,dx$

$$-\cos y + \sin y = 2\left[(\log x)\int x\,dx - \int\left(\frac{d}{dx}\log x \int x\,dx\right)dx\right] + \frac{x^2}{2} + c$$

$$-\cos y + \sin y = \cancel{2}\log x \cdot x^2 - \cancel{\frac{x^2}{2}} + \cancel{\frac{x^2}{2}} + c$$

$$\boxed{\sin y - \cos y = x^2 \log x + c}$$ <u>Ans</u>.

2. $e^x \tan y\, dx = -(1+e^x)\sec^2 y\, dy$

$$\frac{e^x}{1+e^x}\,dx = -\frac{\sec^2 y}{\tan y}\,dy$$

$$\log(1+e^x) = -\log\tan y + \log c \Rightarrow \cancel{\log}(1+e^x).\tan y = \cancel{\log} c$$

$$\boxed{(1+e^x)\tan y = c}$$ <u>Ans</u>.

dx

$$\frac{dy}{1+y} = (1+x)dx$$

$$\boxed{\log(1+y) = x + \frac{x^2}{2} + c}$$ Ans.

16 Dec 2014
L-3

Equations reducible to variable separable :

1) $$\frac{dy}{dx} = \sin(x+y)$$

Let $x+y=t$; Diff. w.r.t. x $\quad 1 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$

$$\frac{dt}{dx} - 1 = \sin t \Rightarrow \int \frac{dt}{1+\sin t} = \int dx$$

$$\int \frac{1-\sin t}{(1-\sin t)(1+\sin t)} dt = x+c \Rightarrow \frac{1-\sin t}{\cos^2 t} dt = x+c$$

$$\int \sec^2 t\, dt - \int \sec t \tan t\, dt = x+c \Rightarrow \tan t - \sec t = x+c$$

$$\boxed{\tan(x+y) - \sec(x+y) = x+c}$$ Ans.

Homogeneous D.E :

$$\frac{dy}{dx} = f(x,y)$$ where $f(x,y)$ is H.f^n of degree '0'.

Put $y = vx$ & solve.

Ex :- $$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$$ R.H.S. is H.f^n of degree '0'.

Put $y = vx$, Diff. w.r.t. x → $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\cancel{v} + x\frac{dv}{dx} = \cancel{v} + \tan v$$

$$\int \frac{dv}{\tan v} = \int \frac{dx}{x} = \log \sin v = \log x + \log c$$

$$\sin v = xc \Rightarrow \boxed{\sin\frac{y}{x} - xc = 0}$$ Ans

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

(i) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ we can reduce it to homogeneous form.

Put $x = X + h$ & $y = Y + k$

$$\boxed{\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}} \rightarrow H.f^n.$$

where h & k are given by

$$a_1h + b_1k + c_1 = 0$$
$$a_2h + b_2k + c_2 = 0$$

Ex :- $\frac{dy}{dx} = \frac{2x + y + 6}{-x + y - 3}$ find h & k which reduces this non H. form to H. form.

Put $x = X + h$ & $y = Y + k$

$$2h + k + 6 = 0 \quad \& \quad -h + k - 3 = 0$$

$$\begin{array}{ccccccc} 1 & ^{h} & 6 & ^{k} & 2 & ^{1} & 1 \\ 1 & & -3 & & -1 & & 1 \end{array}$$

$$\frac{h}{-3-6} = \frac{k}{-6+6} = \frac{1}{2+1}$$

$$\Rightarrow \boxed{h = -3 \ \& \ k = 0} \text{ Ans.}$$

Substitution is $x = X - 3$ &
$y = Y$

Ex :- $\frac{dy}{dx} = \frac{x - y + 2}{x + y - 1}$

$$h - k + 2 = 0$$
$$h + k - 1 = 0$$
$$2h + 1 = 0$$

$$\boxed{h = -1/2 \ \& \ k = 3/2} \text{ Ans.}$$

Exact DE :-

Def :- The Differential form $Mdx + Ndy$ is called exact if there exists $u(x, y)$ such that

$$du = Mdx + Ndy$$

Condition for exactness : $Mdx + Ndy$ exists if & only if (iff)

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

General Sol^n –

$$\underset{y\,\text{const.}}{\int M dx} + \int N dy = C$$

(terms not containing x or terms indipendent of x)

Ex :– $y dx + x dy = 0$

$$\frac{\partial M}{\partial y} = 1 \;,\; \frac{\partial N}{\partial x} = 1 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow \text{Exact}$$

general sol^n :– $\underset{y\,\text{const}}{\int y dx} + \underset{\text{no } x}{\int \cancel{x dy}} = c$

$$\boxed{xy = c} \text{ Ans.}$$

Ex :– $2xy^2 dx + 2x^2 y dy = 0$

$$\frac{\partial M}{\partial y} = 2\cdot 2xy \qquad \frac{\partial N}{\partial x} = 2\cdot 2xy \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \text{Exact}$$

General Sol^n :– $y^2 \underset{y\,\text{const}}{\int 2x dx} + \underset{\text{no } x}{\int \cancel{2x^2 y dy}} = c$

$$\boxed{x^2 y^2 = c} \text{ Ans.}$$

Ex :– $(x^4 - 2xy^2 + y^4) dx - (2x^2 y - 4xy^3 + \sin y) dy = 0$

$$M = x^4 - 2xy^2 + y^4 \;,\; N = -2x^2 y + 4xy^3 - \sin y$$

$$\frac{\partial M}{\partial y} = -4xy + 4y^3 \;,\; \frac{\partial N}{\partial x} = -4xy + 4y^3$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow \text{Exact}$$

Gen. Sol^n :– $\frac{x^5}{5} - x^2 y^2 + xy^4 - \int \sin y\, dy = c$

$$\boxed{\frac{x^5}{5} - x^2 y^2 + xy^4 + \cos y = c} \text{ Ans.}$$

$$\frac{1}{y^2}\left[y\,dx - x\,dy\right] = 0$$

$$\frac{y\,dx - x\,dy}{y^2} = 0 \Rightarrow d\left(\frac{x}{y}\right) = 0 \Rightarrow \boxed{\frac{x}{y} = c}$$

makes non exact form to exact form. It is known as Integrating Factor (IF).

Integrating Factor – A factor which reduces a non-exact D.E. to exact D.E. [After multiplying IF non-exact becomes exact].

Non exact D.E. reducible to exact D.E. –

Methods of Finding IF :—

Let $M\,dx + N\,dy = 0$ is non exact DE.

Form I : M & N are homogeneous f

M & N are homogeneous f^n of same degree

$$I.F. = \frac{1}{Mx + Ny}, \quad Mx + Ny \neq 0 \text{ (provided)}$$

Form 2 : $M = y\,f_1(x,y)$ & $N = x\,f_2(x,y)$

$$I.F. = \frac{1}{Mx - Ny}, \text{ provided } Mx - Ny \neq 0.$$

Ex :— $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

$M = x^2y - 2xy^2$, $N = -x^3 + 3x^2y$

$= y(x^2 - 2xy) \rightarrow \times\, y\, f_1(x,y)$

$$\frac{\partial M}{\partial y} = x^2 - 4xy, \quad \frac{\partial N}{\partial x} = -3x^2 + 6xy \qquad \text{Non-exact}$$

But M & N are H.f^n of same degree (3),

$$I.F. = \frac{1}{Mx + Ny} = \frac{1}{\cancel{x^3y} - 2x^2y^2 - \cancel{x^3y} + 3x^2y^2} = \frac{1}{x^2y^2}$$

$$\frac{x^2y - 2xy^2}{x^2y^2}\,dx - \frac{x^3 - 3x^2y}{x^2y^2}\,dy = 0 \qquad \text{exact.}$$

Gen sol^n :—

$$\underset{y \text{ const.}}{\int\left(\frac{1}{y} - \frac{2}{x}\right)dx} - \underset{\text{no } x}{\int\left(\cancel{\frac{x}{y^2}} - \frac{3}{y}\right)dy} = c \Rightarrow \boxed{\frac{x}{y} - 2\log x + 3\log y = c}$$ Ans

Sol^n :— $M = xy^2 + 2x^2y^3$ $\qquad$ $N = x^2y - x^3y^2$

$= y(xy + 2x^2y^2)$ $\qquad$ $= x(xy - x^2y^2)$

$= y\, f_1(x,y)$ $\qquad$ $= x\, f_2(x,y)$

$$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2 \neq \frac{\partial N}{\partial x} = 2xy - 3x^2y^2 \rightarrow \text{Non-exact}$$

But $M = y f_1(x,y)$ & $N = x f_2(x,y)$

$$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{\cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3} = \frac{1}{3x^3y^3}$$

$$\frac{xy^2 + 2x^2y^3}{3x^3y^3}\,dx + \frac{x^2y - x^3y^2}{3x^3y^3}\,dy = 0 \rightarrow \text{Exact form.}$$

Gen. sol^n :—
$$\underset{y\text{ const.}}{\int}\left(\frac{1}{3x^2y} + \frac{2\times1}{3\;x}\right)dx + \underset{\text{no }x}{\int}\left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = c$$

$$\boxed{\frac{-1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c}\ \underline{\text{Ans}}.$$

Form 3 :
$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \quad [f^n \text{ of } x \text{ alone}]$$

$$\text{I.F.} = e^{\int f(x)\,dx}$$

Form 4 :
$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y) \quad [f^n \text{ of } y \text{ alone}]$$

$$\text{I.F.} = e^{\int g(y)\,dy}$$

Ex :—(1) $(x^2 + y^2 + x)\,dx + xy\,dy = 0$

Sol :— $M = x^2 + y^2 + x$, $N = xy$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = y$ $\qquad$ Non-exact

N is looking simpler so first check by it.

$$\frac{\partial y \qquad \partial x}{N} = \frac{-y \quad y}{xy} = \frac{\quad}{x} = f(x)$$

$$I.F. = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$$x(x^2+y^2+x)\,dx + x(xy)\,dy = 0 \longrightarrow \text{Exact eq}^n$$

Gen sol^n :—

$$\underset{y \text{ const}}{\int}(x^3+xy^2+x^2)\,dx + \underset{\text{no } x}{\int} x^2y\,dy = c$$

$$\boxed{\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c} \quad \underline{\text{Ans.}}$$

Ex :— $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$

Sol^n :— $M = y^4+2y \quad , \quad N = xy^3+2y^4-4x$

$$\frac{\partial M}{\partial y} = 4y^3+2 \quad , \quad \frac{\partial N}{\partial x} = y^3-4 \longrightarrow \text{Non-exact}$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{y^3-4-4y^3-2}{y^4+2y} = \frac{-3y^3-6}{y^4+2y} = g(y) = \frac{-3}{y}$$

$$I.F. = e^{\int g(y)dy} = \frac{1}{y^3}$$

$$\frac{y^4+2y}{y^3}\,dx + \frac{xy^3+2y^4-4x}{y^3}\,dy = 0 \longrightarrow \text{Exact}$$

G. Sol^n :—

$$\underset{y\text{const}}{\int}\left(y + \frac{2}{y^2}\right)dx + \underset{\text{no } x}{\int}\left(\cancel{x} + 2y - \cancel{\frac{4x}{y^3}}\right)dy = c$$

$$\boxed{xy + \frac{2x}{y^2} + y^2 = c} \quad \underline{\text{Ans.}}$$

Q :— If $x^\alpha y^\beta$ is I.F. of $(x^7y^2+3y)\,dx + (3x^8y-x)\,dy = 0$

$\alpha = ?$ $\quad\quad$ $\beta = ?$

$$\left(x^{\alpha+7}y^{\beta+2} + 3x^{\alpha}y^{\beta+1}\right)dx + \left(3x^{\alpha+8}y^{\beta+1} - x^{\alpha+1}y^{\beta}\right)dy = 0 \rightarrow \text{Exact}$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ must be satisfied.

$$(\beta+2)x^{\alpha+7}y^{\beta+1} + (\beta+1)3x^{\alpha}y^{\beta} = (\alpha+8)3x^{\alpha+7}y^{\beta+1} - (\alpha+1)x^{\alpha}y^{\beta})$$

$$3(\alpha+8) = \beta+2 \quad \& \quad 3(\beta+1) = -(\alpha+1)$$

~~$\alpha-\beta = -6$~~ ~~& $\beta+1 = -\alpha-1$~~

~~$\alpha+\beta = -2$~~

~~$\alpha-\beta = -6$~~
~~$\alpha+\beta = -2$~~
~~$2\alpha = -8$~~
~~$\alpha = -4$ & $\beta = 2$~~

$$3\alpha-\beta = -22$$
$$\alpha+3\beta = -4$$
$$10\alpha = -70$$
$$\boxed{\alpha = -7 \ \& \ \beta = 1} \ \text{Ans.}$$

Linear D.E :—

Linear in y : $\frac{dy}{dx} + Py = Q$, where P, Q are f^n's of x

$$IF = e^{\int P dx}$$

General Sol^n:— $\boxed{y \times I.F. = \int Q \cdot IF dx + c}$ *

Linear in x : $\frac{dx}{dy} + Px = Q$, where P, Q are f^ns of y

$$I.F = e^{\int P dy}$$

General Sol^n :— $\boxed{x \times I.F. = \int Q \cdot IF dy + c}$

ex :— $\boxed{e^{\int \frac{a}{x}dx} = x^a}$

$e^{a\log x} = \cancel{e^{\log}} x^a$

Q:— $x^2\frac{dy}{dx} + 2xy = \frac{2\log x}{x}$

Sol^n:— $\frac{dy}{dx} + \frac{2}{x}y = \frac{2\log x}{x^3}$ $\quad P = \frac{2}{x}$, $Q = \frac{2\log x}{x^3}$

(Linear in y)

Gen. solⁿ :- $y.x^2 = \int Q.I.F.\,dx + c = 2\int \log x \cdot \frac{1}{x}\,dx + c$

Let $\log x = t$; $\frac{1}{x}dx = dt$

$yx^2 = \frac{\cancel{2}t^2}{\cancel{2}} + c \Rightarrow \boxed{x^2y = [\log x]^2 + c}$ Ans.

Q:- $(x+2y^3)\frac{dy}{dx} = y$

Solⁿ:- $y\frac{dx}{dy} = x + 2y^3 \Longrightarrow \frac{dx}{dy} = \frac{x}{y} + 2y^2$

$\frac{dx}{dy} - \frac{1}{y}x = 2y^2$ (Linear in x) $P = \frac{-1}{y}$, $Q = 2y^2$, $IF = e^{\int \frac{1}{y}dy} = \frac{1}{y}$

G.S :- $x \cdot IF = \int Q.\,IF\,dy + c$

$\frac{x}{y} = \int 2y^{\cancel{2}} \times \frac{1}{\cancel{y}}\,dy + c \Rightarrow \boxed{\frac{x}{y} = y^2 + c}$ Ans.

Q:- Maxᵐ value of y ; $y' + 2y\tan x = \sin x$; $y(\pi/3) = 0$

Solⁿ:- $\frac{dy}{dx} + (2\tan x)y = \sin x$ ⟶ Linear in y.

$I.F. = e^{\int 2\tan x\,dx} = e^{2\log\sec x} = \sec^2 x$

GS:- $y.I.F = \int Q.I.F\,dx + C$

$y.\sec^2 x = \int \sin x.\sec^2 x\,dx + C = \int \tan x\sec x\,dx + C$

$y\sec^2 x = \sec x + C \Rightarrow \boxed{y = \cos x + C\cos^2 x}$

$y\left(\frac{\pi}{3}\right) = 0 \Rightarrow x = \frac{\pi}{3}, y = 0$

$0 = \frac{1}{2} + C\frac{1}{4} \Rightarrow \boxed{C = -2} \Rightarrow y = \cos x - 2\cos^2 x$

1) $\frac{dy}{dx} = -\sin x + 4\cos x\sin x = -\sin x + 2\sin 2x$

2) $y' = 0 \Rightarrow -\sin x + 4\cos x\sin x = 0$

$\sin x(4\cos x - 1) = 0 \Rightarrow x = 0$ or $\cos x = \frac{1}{4}$

3) $y'' = -\cos x + 4\cos 2x = -\cos x + 4[2\cos^2 x - 1] \Rightarrow y''(0) = 3 > 0$ min ×

$\cos x = \frac{1}{4}$

$$\text{Max}^m \text{ value of } y = y\Big|_{\cos x = \frac{1}{4}} = \frac{1}{4} - 2\left(\frac{1}{4}\right)^2 = \frac{1}{4} - \frac{1}{8} = \frac{1}{8} \text{ Ans.}$$

Non-linear eq^{n}s reducible to linear form :—

Bernoulli's eq^n :— $\dfrac{dy}{dx} + Py = Qy^n$

Method :— $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q$ Put $\boxed{y^{1-n} = t}$

Ex :— $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2y^6$

$$y^{-6}\frac{dy}{dx} + \frac{1}{x}y^{(1-6)=-5} = x^2$$

Put $y^{-5} = t$

$$-5y^{-6}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dt}{dx}$$

$$-\frac{1}{5}\frac{dt}{dx} + \frac{1}{x}t = x^2$$

$$\frac{dt}{dx} - \frac{5}{x}\cdot t = -5x^2$$

$$\text{I.F.} = \frac{1}{x^5}$$

G.S :—

$$t\cdot\frac{1}{x^5} = \int -5\cancel{x^2}\frac{1}{\cancel{x^5}_3}dx + c$$

$$= -5\int\frac{1}{x^3}dx + c$$

$$= \frac{-5}{-2}\cdot x^{-2} + c$$

$$\frac{t}{x^5} = \frac{5}{2}x^{-2} + c$$

$$\boxed{\frac{1}{x^5y^5} = \frac{5}{2}x^{-2} + c} \text{ Ans.}$$

2) $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{x\,2\sin y\cancel{\cos y}}{\cos^2 y} = x^3$$

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$$

Put $\tan y = t$, $\sec^2 y\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$$\frac{dt}{dx} + 2x\cdot t = x^3$$

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

G.S :—

$$t\cdot e^{x^2} = \int x^3e^{x^2} + c$$

Put $x^2 = u \Rightarrow \cancel{2}x\,dx = \dfrac{du}{2}$

$$t\cdot e^{x^2} = \int e^{u}u\frac{du}{2} + c$$

$$t\cdot e^{x^2} = \frac{1}{2}\left[u\cdot e^{u} - 1e^{u}\right] + c$$

$$\tan y\cdot e^{x^2} = \frac{e^{x^2}}{2}\left[x^2 - 1\right] + c$$

$$\boxed{e^{x^2}\tan y = \frac{e^{x^2}}{2}\left[x^2-1\right] + c} \text{ Ans.}$$

which every member of one family cuts every member of other family at right angles are called OT's.

Method :–

1) $F(x, y, c) = 0 \to$ Family of curves

2) Eliminating c, we get $f(x, y, \frac{dy}{dx}) = 0 \to$ D.E. of family.

3) Replace $\frac{dy}{dx} = \frac{-dx}{dy}$, we get $f(x, y, \frac{-dx}{dy}) = 0 \to$ D.E. of OT.

$(\because m_1 m_2 = -1)$

4) Solve: $G(x, y, c) = 0 \longrightarrow$ O.T.

Polar coordinates –

Method is same except replace $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$

Ex :– 1) $y = K(x-1)$ 2) $y^2 = 4a(x+a)$ 3) $r = a(1+\cos\theta)$. Find OT.

Sol^n :– $\hookrightarrow$ Family

1) $y' = K \Rightarrow y = y'(x-1) \to$ D.E. of family

Replace $\frac{dy}{dx} = \frac{-1}{y'} \Rightarrow y = \frac{-1}{y'}(x-1) \Rightarrow \boxed{yy' = (1-x)} \to$ D.E. of OT.

$y\frac{dy}{dx} = (1-x) \Rightarrow y\,dy = (1-x)\,dx \Rightarrow \boxed{\frac{y^2}{2} = x - \frac{x^2}{2} + c} \to$ O.T. Ans.

2) $y^2 = 4a(x+a) \to$ Family

$2yy' = 4a \Rightarrow y^{\cancel{2}} = 2\cancel{y}y'(x + \frac{yy'}{2}) \Rightarrow \frac{y}{2y'} = x + \frac{yy'}{2} \to$ D.E of family

Replace $y' = \frac{-1}{y'} \Rightarrow \frac{-yy'}{2} = x - \frac{y}{2y'} \Rightarrow \frac{y}{2y'} = x + \frac{yy'}{2} \to$ D.E. of OT.

$\boxed{\text{D.E. of family} = \text{D.E. of O.T.}} \Rightarrow \boxed{\text{Family} = \text{OT}}$

These are called as self orthogonal.

→ It means every member of the family cuts each other.

$$\frac{dr}{d\theta} = -a\sin\theta \Rightarrow \frac{\frac{dr}{d\theta}}{r} = \frac{-\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)}$$

$$\boxed{\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}} \rightarrow \text{DE of family}$$

$$\text{Replace } \frac{dr}{d\theta} = -r^2\frac{d\theta}{dr} \Rightarrow r\frac{d\theta}{dr} = \frac{\sin\theta}{1+\cos\theta} \rightarrow \text{DE of OT}$$

Var. S :-

$$\frac{1+\cos\theta}{\sin\theta}\,d\theta = \frac{dr}{r} \Rightarrow \int\frac{\cancel{2}\cos^2\theta/2}{\cancel{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\,d\theta = \log r + \log c$$

$$2\int\cot\frac{\theta}{2}\,d\theta = \log r + \log c = 2\log\sin\frac{\theta}{2} = \log cr$$

$$\Rightarrow \sin^2\frac{\theta}{2} = cr \Rightarrow cr = \frac{1-\cos\theta}{2}$$

$$r = \frac{1}{2c}(1-\cos\theta)$$

$$\boxed{r = b(1-\cos\theta)} \rightarrow \text{OT}$$

Ans.

Linear Differential Eq^{n}s (LDE) –

General form of n^{th} order LDE

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \text{------} \; a_ny = X \rightarrow f^n \text{ of } x \quad \text{——} ①$$

$$[a_0D^n + a_1D^{n-1} + \text{--------} \; a_n]y = X$$

1. $X \equiv 0$ → Homogeneous LDE (HLDE)
2. $X \not\equiv 0$ → Non H. LDE (NHLDE)

have general method to solve this.

a_0, a_1, a_2 ---- a_n are all constant → $\boxed{\text{LDE with const. coefficients}}$

otherwise ① is L.D.E. with variable coefficients.

$$[a_0D^n + a_1D^{n-1} + \text{------} a_n]\,y = X$$

$$\boxed{F(D)\cdot y = X}$$

HLDE : $\boxed{F(D)\cdot y = 0}$

2nd order HLDE : $[a_0D^2 + a_1D + a_2]\,y = 0$

Auxiliary eqn (A.E.) :— $a_0m^2 + a_1m + a_2 = 0$, $\Rightarrow m_{1,2} = \_\_,\_\_$.

Roots can be →

Roots	(LI) Linearly Independent Solns	General Soln
1. Real & distinct (b_1 & b_2)	e^{b_1x}, e^{b_2x}	$c_1e^{b_1x} + c_2e^{b_2x}$
2. Real & equal (b, & b)	e^{bx}, xe^{bx}	$c_1e^{bx} + c_2xe^{bx}$ $(c_1+c_2x)e^{bx}$
3. Complex conjugates ($a \pm ib$)	$e^{ax}\cos bx$, $e^{ax}\sin bx$	$e^{ax}(c_1\cos bx + c_2\sin bx)$

Ex:—

	Roots	Gen. Soln
1)	$1, 2$	$c_1e^x + c_2e^{2x}$
2)	± 1	$c_1e^x + c_2e^{-x}$
3)	$2, 2$	$(c_1 + c_2x)e^{2x}$
4)	$2, 2, 2$	$(c_1 + c_2x + c_3x^2)e^{2x}$
5)	$1 \pm i$	$e^x[c_1\cos x + c_2\sin x]$
6)	$\frac{1 \pm i\sqrt{3}}{2}$	$e^{x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right]$
7)	$\pm 2i$	$c_1\cos 2x + c_2\sin 2x$
8)	$1, 2, 3$	$c_1e^x + c_2e^{2x} + c_3e^{3x}$
9)	$2, 3, 3$	$c_1e^{2x} + (c_2 + c_3x)e^{3x}$
10)	$2, 1 \pm i$	$c_1e^{2x} + e^x[c_2\cos x + c_3\sin x]$

dx^2 dx $\qquad \qquad \frac{\quad}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

3. $\frac{d^2y}{dx^2} + 9y = 0$ 4. $\frac{d^2y}{dx^2} - 9y = 0$ 5. $y''' - y = 0$

6. $y'' + y' + y = 0$ 7. $[D^4 + D^2 + 1]y = 0$

Solⁿ:—

① A.E.: $m^2 - 5m + 6 = 0 \Rightarrow m = 2, 3$

$y = C_1 e^{2x} + C_2 e^{3x}$

② $(D^2 - 4D + 4)y = 0 \Rightarrow$ A.E: $m^2 - 4m + 4 = 0$ $m = 2, 2$

$y = (C_1 + C_2 x)e^{2x}$

③ $(D^2 + 9)y = 0 \Rightarrow$ A.E: $m^2 + 9 = 0 \Rightarrow m = \pm i3$

$y = C_1 \cos 3x + C_2 \sin 3x$

④ $(D^2 - 9)y = 0 \Rightarrow$ A.E: $m^2 - 9 = 0 \Rightarrow m^2 = 9 \Rightarrow m = \pm 3$

$y = C_1 e^{3x} + C_2 e^{-3x}$

⑤ $(D^3 - 1)y = 0 \Rightarrow$ A.E: $m^3 - 1 = 0 \Rightarrow \cancel{m^3 = 1 \Rightarrow m = 1, 1, 1}$

$(m-1)(m^2 + m + 1) = 0$

$m = 1, \ m = \frac{-1 \pm \sqrt{1-4}}{2}$

$y = (C_1 e^{x} \cancel{C_2 x + C_3 x^2) e^{x}}$

$+ e^{-x/2}\left[C_2 \cos \frac{\sqrt{3}}{2}x + C_3 \sin \frac{\sqrt{3}}{2}x\right]$

⑥ $(D^2 + D + 1)y = 0 \Rightarrow$ A.E: $m^2 + m + 1 = 0 \Rightarrow m = \frac{-1 \pm \sqrt{1-4}}{2}$

$m = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$

$y = e^{-x/2}\left[C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x\right]$

⑦ A.E: $m^4 + m^2 + 1 = 0 \Rightarrow m^4 + m^2 + m^2 + 1 - m^2 = 0$

$m^4 + 2m^2 + 1 - m^2 = 0$

$(m^2 + 1)^2 - m^2 = 0$

$(m^2 + 1 - m)(m^2 + 1 + m) = 0$

$m^2 + 1 - m = 0$ & $m^2 + 1 + m = 0$

$m = \frac{1 \pm \sqrt{1-4}}{2}$; $m = \frac{-1 \pm \sqrt{1-4}}{2}$

$m = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$; $m = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$

$y = \cancel{C_1 e^{x/2} + C_2 e}$

$e^{x/2}\left[C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x\right]$

$+ e^{-x/2}\left[C_3 \cos \frac{\sqrt{3}}{2}x + C_4 \sin \frac{\sqrt{3}}{2}x\right]$

Sol^n :- $(D^3 - 9D^2 + 23D - 15)y = 0$

A.E. $= m^3 - 9m^2 + 23m - \underline{15} = 0$

↳ Roots are factors of constants

$$\begin{array}{c|cccc|}
1 & 1 & -9 & 23 & -15 \\
 & +0 & +1 & \mp 8 & 15 \\
\hline
 & 1 & -8 & 15 & \boxed{0}
\end{array}$$

$0 \longrightarrow$ If zero comes here 1 is factor. (or any other)

$(m-1)(m^2 - 8m + 15) = 0 \Rightarrow m = 1, 3, 5$

$y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{5x}$

⑨ $x'' + 6x' + 8x = 0$; $x(0) = 1$; $x'(0) = 0$

Sol^n:- If D.E. is in x then independent var. is 't'.

$(D^2 + 6D + 8)x = 0 \Rightarrow$ A.E: $m^2 + 6m + 8 = 0 \Rightarrow m = -2, -4$

$x = c_1 e^{-2t} + c_2 e^{-4t}$

$x(0) = 1 \Rightarrow 1 = c_1 + c_2$

$x'(0) = 0 \Rightarrow x' = -2c_1 e^{-2t} - 4c_2 e^{-4t}$

$0 = -2c_1 - 4c_2 \Rightarrow c_1 + 2c_2 = 0$

$c_1 + c_2 = 1$

$c_1 + 2c_2 = 0$

$-c_2 = 1$

$c_2 = -1$ & $c_1 = 2$

$x = 2e^{-2t} - e^{-4t}$ Ans.

⑩ $x'' + 9x = 0$; $x(0) = 1$, $x'(0) = 1$

Sol^n:- $(D^2 + 9)x = 0 \Rightarrow$ A.E : $m^2 + 9 = 0 \Rightarrow m = \pm i\cancel{\sqrt{}}3$

$x = c_1 \cos \cancel{\sqrt{}}3t + c_2 \sin \cancel{\sqrt{}}3t$

$x(0) = 1 \Rightarrow 1 = c_1 + 0 \Rightarrow c_1 = 1$

$x'(0) = 1 \Rightarrow x' = -c_1 \sin \cancel{\sqrt{}}3t + c_2 \cos \cancel{\sqrt{}}3t$

$1 = 0 + 3c_2 \Rightarrow c_2 = 1/3$

$x = \cos \cancel{\sqrt{}}3t + \frac{1}{3} \sin \cancel{\sqrt{}}3t$ Ans.

General Sol^n : $\boxed{y = y_c + y_p}$ *

where y_c = Complementary f^n (C.F.) : It is sol^n of corresponding $F(D)y = 0$.

→ having constants

y_p = Particular Integral (P.I.) or Particular Sol^n.

= Sol^n of $F(D)y = X$ → ~~not~~

→ Not having any const.

$$\boxed{y_p = \frac{1}{F(D)} X}$$

Note :– 1) $\frac{1}{D}$ means Integration

ex $\frac{1}{D} \sin 2x = \frac{-\cos 2x}{2}$

2) $\boxed{\frac{1}{D-a} X = e^{ax} \int X e^{-ax}\, dx}$

ex :– $\frac{1}{D-2} e^{5x} = e^{2x} \int e^{5x} e^{-2x}\, dx = e^{2x} \cdot \int e^{3x} dx = e^{2x} \cdot \frac{e^{3x}}{3}$

$= \frac{e^{5x}}{3}$

Shortcut : $\frac{1}{D-2} e^{\textcircled{5}x}$ → Put D = 5 → $\frac{e^{5x}}{3}$

Methods of finding PI :—

1) $X = e^{ax} \Rightarrow y_p = \frac{1}{F(D)} e^{ax}$

Put D = a

$y_p = \frac{1}{F(a)} e^{ax}$, if $F(a) \neq 0$

If $F(a) = 0$; $y_p = x \left[\frac{1}{F'(D)} e^{ax} \right]$

Put D = a ; $y_p = x \left[\frac{1}{F'(a)} e^{ax} \right]$, if $F'(a) \neq 0$

$$y_p = \frac{1}{F(D)}\sin ax = \frac{1}{\phi(D^2, D)}\sin ax$$

Put $D^2 = -a^2$

$$y_p = \frac{1}{\phi(-a^2, D)}\sin ax \quad \text{if} \quad \phi(-a^2 . D) \neq 0$$

If $\phi(-a^2 . D) = 0 \Rightarrow y_p = x\left[\frac{1}{\phi'(D^2, D)}\sin ax\right]$

Put $D^2 = -a^2$ if $\phi'(D^2, D) \neq 0$.

& so on.

Ex :- $y'' - 3y' + 2y = 5e^{4x}$

Sol^n :- $(D^2 - 3D + 2)y = 5e^{4x}$

gen. sol^n : $y_c + y_p$

For y_c : A.E. $= m^2 - 3D + 2 = 0 \Rightarrow m = 1, 2$

$y_c = c_1 e^{x} + c_2 e^{2x}$

For y_p : $y_p = \frac{1}{D^2 - 3D + 2} 5e^{4x}$; Put $D = 4$

$$y_p = \frac{1}{6} 5e^{4x} = \frac{5}{6}e^{4x}.$$

gen sol^n :- $y = c_1 e^{x} + c_2 e^{2x} + \frac{5}{6}e^{4x}$ Ans.

2) $y'' - 4y' + 4y = 3e^{2x}$

Sol^n :- $(D^2 - 4D + 4)y = 3e^{2x}$

y_c : A.E. : $m^2 - 4m + 4 = 0$; $m = 2, 2$

$y_c = [c_1 + c_2 x]e^{2x}$

y_p : $y_p = \frac{1}{D^2 - 4D + 4} 3e^{2x}$

Put $D = 2$, Denominator $(D_r) = 0$

$$= 3x\left[\frac{1}{2D - 4}e^{2x}\right]$$

Put $D = 2$, $D_r = 0$

$$y_p = \frac{3x^2e^{2x}}{2}$$

Gen solⁿ $= y_c + y_p = (C_1 + C_2x)e^{2x} + \frac{3x^2e^{2x}}{2}$ Ans.

3. $y'' - 5y' + 6y = \log 2$

Solⁿ:— $[D^2 - 5D + 6]y = \log 2$

$$y_p = \frac{1}{D^2-5D+6} \log 2 \cdot e^{0x}$$

Put $D=0$; $y_p = \frac{\log 2}{6}$

Q:- 1. $y'' + 2y' + y = \sin 2x$

Solⁿ:— $[D^2 + 2D + 1]y = \sin 2x$

y_c: A.E. : $m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$

$$y_c = [C_1 + C_2x]e^{-x}$$

$$y_p = \frac{1}{[D^2+2D+1]} \sin 2x$$

Put $D^2 = -2^2 = -4 \Rightarrow y_p = \frac{1}{-4+2D+1} \sin 2x = \frac{1}{2D-3} \sin 2x$

$y_p = \frac{2D+3}{4D^2-9} \sin 2x$; Put $D^2 = -4$

$$y_p = \frac{2D+3}{-16-9} \sin 2x = \frac{2D+3}{-25} \sin 2x = \frac{-1}{25}\left[2D\sin 2x + 3\sin 2x\right]$$

$$y_p = \frac{-1}{25}\left[4\cos 2x + 3\sin 2x\right]$$

Gen. Solⁿ: $y = [C_1 + C_2x]e^{-x} - \frac{1}{25}\left[4\cos 2x + 3\sin 2x\right]$ Ans.

2. $[D^2 + 4]y = \cos 2x$

Solⁿ:— y_c: $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$$y_c = C_1\cos 2x + C_2\sin 2x$$

D +4

Put $D^2 = -4$, $D\gamma = 0$

$$y_p = x\left[\frac{1}{\underset{\hookrightarrow \int}{2D \cancel{+4}}}\cos 2x\right] = \frac{x\sin 2x}{4}$$

$$y = C_1\cos 2x + C_2\sin 2x + \frac{x\sin 2x}{4}$$ Ans.

Formulas :—

1) $$\boxed{\frac{1}{(D-a)^r}e^{ax} = \frac{x^r}{r!}e^{ax}}$$

Ex :— $$\frac{1}{(D-2)^2}e^{2x} \Rightarrow \frac{x^2}{2!}e^{2x}$$

2) $$\boxed{\frac{1}{D^2+a^2}\sin ax = \frac{-x}{2a}\cos ax}$$

3) $$\boxed{\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax}$$

Ex :— $$\frac{1}{D^2+4}\cos 2x = \frac{x}{4}\sin 2x$$

$\boxed{\begin{array}{c}\text{17 Dec 2014}\\ \text{L-4}\end{array}}$

Ex :— $D^2x^2 = 2$ $D^4x^2 = 0$

$D^3x^2 = 0$ $\vdots$

Binomial Expansions :—

$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots\ldots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$$

Form 3 : $X = x^n$

$$y_p = \frac{1}{F(D)}x^n = \frac{1}{(1+\phi(D))^n}x^n = (1+\phi(D))^{-n}x^n$$

y_c : $\quad AE = m^2+2m+1 = 0 \quad \Rightarrow \quad m = -1, -1$

$$y_c = [C_1 + C_2x]e^{-x}$$

y_p :
$$y_p = \frac{1}{D^2+2D+1} \cdot x^2 = \frac{1}{[1+(D^2+2D)]^{1}} x^2$$

$$y_p = [1+(D^2+2D)^{-1}]x^2 = [1-(D^2+2D)+(D^2+2D)^2]x^2$$

[Higher order terms vanish.

$$y_p = [1 - D^2 - 2D + \cancel{D^4} + \cancel{4D^3} + 4D^2]x^2$$

$$= x^2 - 2 - 4x + 8 \quad = x^2 - 4x + 6$$

(or) $$y_p = \frac{1}{(D+1)^2} x^2 = [1+D]^{-2} x^2 = [1 - 2D + 3D^2]x^2$$
$$= x^2 - 4x + 6$$

Form 4 : $X = e^{ax} \cdot V$, V is a f^n of x.

$$y_p = \boxed{\frac{1}{F(D)} e^{ax} \cdot V = e^{ax}\left[\frac{1}{F(D+a)} \cdot V\right]}$$

Form 5 : $X = x \cdot V$, V is a f^n of x.

$$y_p = \boxed{\frac{1}{F(D)} x \cdot V = x \frac{1}{F(D)} V - \frac{F'(D)}{[F(D)]^2} V}$$

Ex :- $(D^2+2D+1)y = e^x \cdot \sin 2x$

$$y_p = \frac{1}{D^2+2D+1} e^x \cdot \sin 2x = e^x \left[\frac{1}{(D+1)^2+2(D+1)+1}\right] \sin 2x$$

$$= e^x \left[\frac{1}{D^2+4D+4} \sin 2x\right] = e^x \left[\frac{1}{\cancel{-4}+4D+\cancel{4}} \sin 2x\right]$$

$$= -\frac{e^x}{4} \frac{\cos 2x}{2} = -\frac{e^x \cos 2x}{8} \quad \underline{\text{Ans}}.$$

$$(D^2+9)y = x\cdot\sin x$$

$$y_p = \frac{1}{D^2+9}\left[x\sin x\right] = x\frac{1}{D^2+9}\sin x - \frac{2D}{(D^2+9)^2}\sin x$$

$$= x\frac{1}{-1+9}\sin x - \frac{2D}{(-1+9)^2}\sin x = \frac{x}{8}\sin x - \frac{1}{32}\cos x \quad \underline{\underline{\text{Ans}}}.$$

Ex :- Find y_p : $y'' - 2y' + y = xe^x\sin x$

<u>Sol^n</u>:- $(D^2 - 2D+1)y = xe^x\sin x$

$$y_p = \frac{1}{D^2-2D+1}e^x(x\sin x) = e^x\left[\frac{1}{(D+1)^2-2(D+1)+1}x\sin x\right]$$

$$y_p = e^x\left[\frac{1}{D^2}(x\sin x)\right] = e^x\left[x\frac{1}{D^2}\sin x - \frac{(D^2)'}{(D^2)^2}\sin x\right]$$

$$= e^x\left[-x\sin x - \frac{2D}{(-1)^2}\sin x\right] = e^x\left[-x\sin x - 2\cos x\right]$$

<u>L·D·E· with variable coefficients</u> – reducible to L·D·E· with constant coefficients.

1) Cauchy – Euler Homogeneous L·D·E· –

Homogeneous

$$\boxed{a_0x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_2y = X}$$

$$\left[a_0x^2D^2 + a_1xD + a_2\right]y = X$$

Put $x = e^z$ (or) $z = \log x$

$xD \equiv D_1$; $D = \frac{d}{dx}$, $D_1 = \frac{d}{dz}$

$x^2D^2 = D_1(D_1-1)$

$x^3D^3 = D_1(D_1-1)(D_1-2)$

⋮

Lengendre's L·D·E· –

$$C_0(ax+b)^2\frac{d^2y}{dx^2} + C_1(ax+b)\frac{dy}{dx} + C_2y = X$$

Put $ax+b = e^z$ (or) $z = \log(ax+b)$

$D=\frac{d}{dx}$ & $D_1=\frac{d}{dz}$

$$(ax+b)D = aD_1$$

$$(ax+b)^2D^2 = a^2D_1(D_1-1)$$

$$(ax+b)^3D^3 = a^3D_1(D_1-1)(D_1-2)$$

$$\vdots$$

Q:- 1. $[x^2D^2 - 4xD + 6]y = x^2$

$x = e^z$ or $z = \log x$

$xD = D_1$ & $x^2D^2 = D_1(D_1-1)$; $D_1 = \frac{d}{dz}$

$$[D_1(D_1-1) - 4D_1 + 6]y = (e^z)^2$$

$$[D_1^2 - 5D_1 + 6]y = e^{2z}$$

$y_c = m^2 - 5m + 6$; $m = 2,3$

$$y_c = c_1e^{2z} + c_2e^{3z}$$

$$y_p = \frac{1}{D_1^2-5D_1+6}e^{2z}$$

Put $D_1 = 2$, $Dr = 0$

$$y_p = z\frac{1}{2D_1-5}e^{2z}$$

Put $D_1 = 2$

$$y_p = -ze^{2z}$$

$$y = y_c + y_p$$

$$= C_1e^{2z} + C_2e^{3z} - ze^{2z}$$

$$= C_1e^{2\log x} + C_2e^{3\log x} - \log x\, e^{2\log x}$$

$$\boxed{y = C_1x^2 + C_2x^3 - x^2\log x}$$

2. $(2x-1)^2y'' + (2x-1)y' - 2y = 0$

Put $(2x-1) = e^z$ or $z = \log(2x-1)$

$$(2x-1)D = 2D_1$$

$$(2x-1)^2D^2 = 4D_1(D_1-1)$$

$D_1 = \frac{d}{dz}$

$$[4D_1(D_1-1) + 2D_1 - 2]y = 0$$

$$(4D_1^2 - 2D_1 - 2)y = 0$$

$$4m^2 - 2m - 2 = 0$$

or

$$2m^2 - 1 - 1 = 0$$

$$m = \frac{1\pm3}{4} = 1, \frac{-1}{2}$$

$$y = C_1e^z + C_2e^{-1/2z}$$

$$= C_1e^{\log(2x-1)} + C_2e^{\log(2x-1)^{-1/2}}$$

$$\boxed{y = C_1(2x-1) + C_2(2x-1)^{-1/2}}$$

$\begin{vmatrix} 4 & 6 \end{vmatrix} = 12 - 12 = 0$ $\quad\quad$ $\begin{vmatrix} 5 & 2 \end{vmatrix} = 6 - 5 = 1$

Wronskian :—

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Result :- If $W(y_1, y_2) \neq 0$ then y_1 & y_2 are linearly Independent (L.I.)

If $W(y_1, y_2) = 0$ then y_1 & y_2 are linearly dependent (LD)

Ex :- $y_1 = 2x \quad y_2 = 3x$

$$W = \begin{vmatrix} 2x & 3x \\ 2 & 3 \end{vmatrix} = 6x - 6x = 0 \Rightarrow \text{L.D.} \qquad y_2 = \frac{3}{2} y_1$$

↳ We can express one in terms of other one.

Ex :- $y_1 = e^{2x}$, $y_2 = e^{3x}$

$$W = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} \neq 0 \quad \text{L.I.}$$

Wronskian of three f^ns: $W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$

Method of variation of parameters :—

$$a_0 y'' + a_1 y' + a_2 y = X$$

Let $\boxed{y_c = C_1 y_1 + C_2 y_2}$

$$\boxed{y_p = y_1 \int \frac{-y_2}{W} . X \, dx + y_2 \int \frac{y_1}{W} . X \, dx}$$

y_c : AE : $m^2+4=0 \Rightarrow m = \pm 2i$

$$\boxed{y_c = C_1 \cos 2x + C_2 \sin 2x}$$

$y_1 = \cos 2x$; $y_2 = \sin 2x$

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\left[\cos^2 2x + \sin^2 2x\right] = 2$$

By method of variation of parameters :—

$$y_p = y_1 \int \frac{-y_2}{W} \sec 2x \, dx + y_2 \int \frac{y_1}{W} \sec 2x \, dx$$

$$= \cos 2x \times \frac{2}{2} \int \frac{-\sin 2x}{2} \sec 2x \, dx + \sin 2x \int \frac{\cos 2x}{2} \sec 2x \, dx$$

$$= \frac{\cos 2x}{2.2} \int \frac{-2\sin 2x}{\cos 2x} dx + \frac{\sin 2x}{2} \int dx$$

$$\boxed{y_p = \frac{\cos 2x}{4} \log(\cos 2x) + \frac{x \cdot \sin 2x}{2}}$$

LINEAR ALGEBRA

OR

MATRICES

Matrix – Rectangular array of no.s.

$A = [a_{ij}]_{m \times n}$; a_{ij} = i^{th} row j^{th} column element

$m \times n \rightarrow$ Order or size

$m \longrightarrow$ no. of rows , $n \rightarrow$ no. of columns

Row matrix – $[a_{ij}]_{1 \times n}$
(Row Vector)

Ex :– $[1 \;\; 2 \;\; 3]_{1 \times 3}$

Column Matrix – $[a_{ij}]_{m \times 1}$
(Column Vector)

Ex :– $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3 \times 1}$

Matrix Multiplication – $A = [a_{ij}]_{m\times \textcircled{n}} \longleftrightarrow B = [b_{ij}]_{\textcircled{n}\times r}$

$=$

No. of columns of A = No. of rows of B

$C = AB = [C_{ij}]_{m\times r}$

$$C_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{in}b_{nj}$$

1) No. of multiplications reqd. to find $A_{m\times n} \times B_{n\times r} = m\times r\times n$

2) No. of additions reqd. to find $A_{m\times n} \times B_{m\times n} = m\times r\times (n-1)$

1) $A_{m\times n} B_{n\times r} = C_{m\times r}$

To find 1 element $\longrightarrow$ n multiplication

-||- -||- $m\times r$ -||- $\longrightarrow$ $m\times r\times n$ -||-

2) To find 1 element $\longrightarrow$ $n-1$ additions

-||- $m\times r$ -||- $\longrightarrow$ $m * r * (n-1)$

Ex :– No. of multiplications reqd $A_{4\times 5} * B_{5\times 6} = 4\times 6\times 5 = 120$ Ans.

Ex :– $A = [a_{ij}]_{m\times n}$ where $a_{ij}\ (i+j)\ \forall\ i,j$

Sum of elements of A = ______

Soln :–
$$A = \begin{bmatrix} 1+1 & 1+2 & \cdots & 1+n \\ 2+1 & 2+2 & \cdots & 2+n \\ \vdots & \vdots & & \vdots \\ m+1 & m+2 & \cdots & m+n \end{bmatrix}$$

1st row $\rightarrow$ $(1+1+\cdots n\text{ times}) + (1+2+3\cdots n) = n + \frac{n(n+1)}{2}$

2nd row $\rightarrow$ $(2+2+\cdots n\text{ times}) + (1+2+\cdots n) = 2n + \frac{n(n+1)}{2}$

mth row $\rightarrow$ $(m+m+\cdots n\text{ times}) + (1+2+\cdots n) = mn + \frac{n(n+1)}{2}$

$$= n\,\frac{m(m+1)}{2} + mn\left(\frac{n+1}{2}\right)$$

$$= \frac{mn}{2}\left[m+n+2\right] \quad \underline{\text{Ans}}.$$

Ex :- No. of different matrices with 4 elements whose elements can be either 0, 1, 2 or 3.

Sol^n :- $\begin{bmatrix} \square & \square & \square & \square \end{bmatrix}_{1\times 4}$ $\downarrow 4\times4\times4\times4$ or $\begin{bmatrix} \overset{\nearrow 4}{\square} & \overset{\nearrow 4}{\square} \\ \underset{\swarrow 4}{\square} & \underset{\downarrow 4}{\square} \end{bmatrix}_{2\times 2}$ (or) $\begin{bmatrix} \square \rightarrow 4 \text{ ways} \\ \square \rightarrow 4 \\ \square \rightarrow 4 \\ \square \rightarrow 4 \end{bmatrix}_{4\times 1}$

Each position can be filled in 4 ways $\quad 4^4 \qquad\qquad 4^4 \qquad\qquad 4^4$

$$\longrightarrow 4^4 + 4^4 + 4^4 \quad \underline{\text{Ans}}.$$

Elementary Row Transformation (operations) -

1) $R_i \longleftrightarrow R_j$ (Interchanging two rows)

2) $R_i \longrightarrow kR_i \ (k \neq 0)$ [Multiply a row with non-zero const.]

3) $R_i \longrightarrow R_i + kR_j$ [Adding const multiple of other row]

Ex :- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 6 & 7 \end{bmatrix}$

$R_2 \longleftrightarrow R_3$ $\quad\sim \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 7 \\ 2 & 4 & 5 \end{bmatrix}$

$R_2 \rightarrow 2R_2$ $\quad \begin{bmatrix} 1 & 2 & 3 \\ 6 & 12 & 14 \\ 2 & 4 & 5 \end{bmatrix}$

$R_2 \rightarrow R_2 - 3R_3$ $\quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 2 & 4 & 5 \end{bmatrix}$

Leading non-zero element in a row – is the 1st non-zero element in the row.

$$\begin{bmatrix} 0 & \textcircled{2} & 7 & 4 \\ 0 & \textcircled{7} & 9 & 1 \\ 0 & \textcircled{3} & 10 & 5 \end{bmatrix}$$

(or row echelon form)
Echelon Form – A matrix 'A' is said to be in echelon form if 1) Zero rows (if any) must be at the bottom of the matrix.

right of leading non-zero entry in preceding row.

ex:- $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Echelon form
Reduced row echelon form. (RREF)

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Echelon Form

$\begin{bmatrix} \textcircled{5} & 7 & 6 \\ 0 & \textcircled{2} & 3 \\ 0 & \textcircled{1} & 5 \end{bmatrix}$ not echelon form

$\begin{bmatrix} 0 & \textcircled{1} & 2 & 0 & 5 \\ 0 & 0 & 0 & \textcircled{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Echelon Form

$\begin{bmatrix} \textcircled{1} & 2 & 3 \\ 0 & \textcircled{1} & 5 \\ 0 & 0 & \textcircled{6} \end{bmatrix}$ Echelon form

$\begin{bmatrix} 0 & \textcircled{5} & 6 & 7 & 8 \\ 0 & 0 & \textcircled{7} & 8 & 1 \\ 0 & \textcircled{5} & 2 & 1 & 3 \end{bmatrix}$ Not echelon form

Reduced row echelon form — A matrix 'A' is said to be in reduced rowechelon form, if

(1-2) A is echelon form.

3) The leading non-zero entry in any row(if exist) must be 1 (Leading 1).

4) Leading 1 (if exist) is the only non-zero entry in its column.

Ex:- $\begin{bmatrix} \textcircled{1} & 0 & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} \textcircled{1} & 2 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$

↳ only non-zero entry

↑ ↑ ↑ only non-zero entry.

Result – Every matrix can be reduced to
1) Echelon form (Gaussian elimination)
& 2) Reduced row echelon form (Gauss-Jordan elimination)
Using <u>elementary row transformations</u>.

1) Gaussian elimination ⟶ Forward elimination

2) Gauss Jordan –"– ⟶ –"– –"– + Backward elimination

<u>Note</u>:— Before applying elimination techniques –

1) Transfer all zero rows to the bottom of the matrix.

2) Arrange rows in such a way that the row having more zeroes before leading non-zero element is below the row having less zero rows before leading non-zero element.

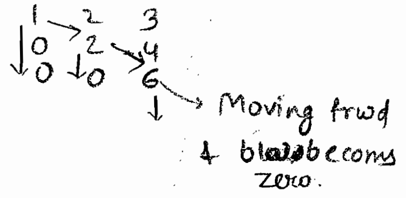
$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ Reduce it to echelon form $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rank – The no. of non-zero rows in echelon form of matrix A is rank of A.

Notation – Rank (A) or $\rho(A)$

Gaussian elimination –

Forward Elimination



Ex :– $A = \begin{bmatrix} \textcircled{1} & 2 & 3 \\ \boxed{2} & 4 & 5 \\ \boxed{3} & 2 & 4 \end{bmatrix}$

Using these make these to 0

$R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 3R_1$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -4 & -5 \end{bmatrix} \quad \xrightarrow{R_2 \leftrightarrow R_3} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 0 & -1 \end{bmatrix}$

$\rho(A) = 3$

Echelon Form

Backward Elimination :–

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 0 & -1 \end{bmatrix} \quad \xrightarrow{R_3 \to R_3/-1} \sim \begin{bmatrix} 1 & 2 & \boxed{3} \\ 0 & -4 & \boxed{-5} \\ 0 & 0 & \textcircled{1} \end{bmatrix} = 0$

Using these

$R_1 \to R_1 - 3R_3$, $R_2 \to R_2 + 5R_3$

$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

~~$R_1 \to R_1 + R_2/2$~~ & $R_2 \to R_2/-4$

$\begin{bmatrix} 1 & \boxed{2} & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_1 \to R_1 - 2R_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Reduced Row echelon form

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ Equal Rows

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 1$

Q:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ $\quad \rho(A) = ?$

$R_2 = 2R_1$ & $R_3 = 3R_1$ (Proportional Rows)

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \rho(A) = 1$

Q:- $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$ $\quad \rho(A)$

$R_3 = 2R_2$, $R_4 = 3R_2$ $\quad A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \to R_2 - R_1$ $\quad \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad \rho(A) = 2$

Q:- $A = \begin{bmatrix} 1 & a & a^2 & \cdots & a^n \\ 1 & a & a^2 & \cdots & a^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a & a^2 & \cdots & a^n \end{bmatrix}$ $\quad \rho(A) = 1$ $\quad \sim \begin{bmatrix} 1 & a & a^2 & \cdots & a^n \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$

Q:- $A = [a_{ij}]_{m \times n}$ $\quad a_{ij} = 5 \ \forall \ i,j$, $\rho(A) =$ ____

$A = \begin{bmatrix} 5 & 5 & \cdots & 5 \\ 5 & 5 & \cdots & 5 \\ \vdots & \vdots & & \vdots \\ 5 & 5 & \cdots & 5 \end{bmatrix} = \begin{bmatrix} 5 & 5 & \cdots & 5 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ $\quad \rho(A) = 1$

Q:- $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 1 & 0 & 5 \\ 0 & 1 & 3 & 2 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ $\quad \rho(A) =$ ____

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & \textcircled{-3} & -2 & -3 \\ 0 & \boxed{1} & 3 & 2 \\ 0 & \boxed{0} & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & \textcircled{7} & 3 \\ 0 & 0 & \boxed{2} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

$\rho(A) = 4$

Transpose –

$A = [a_{ij}]_{m\times n}$, $A^T = [b_{ij}]_{n\times m}$ $\boxed{b_{ij} = a_{ji}}$ $\forall\ i,j$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & -7 \end{bmatrix}_{3\times 2} \qquad A^T = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & -7 \end{bmatrix}_{2\times 3}$$

Results –

1) $(A^T)^T = A$

2) $(A+B)^T = A^T + B^T$

3) $(KA)^T = KA^T$

4) $(AB)^T = B^T A^T$

Square Matrices :–

$A = [a_{ij}]_{n\times n} \to$ Square matrix of order n.

Diagonal –

$A = [a_{ij}]_{n\times n}$ where $a_{ij} = 0 \ \forall\ i \neq j$.

ex :– $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Upper Triangular Matrix –

$A = [a_{ij}]_{n\times n}$ where $a_{ij} = 0 \ \forall\ i > j$

Ex:– $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

$A = [a_{ij}]_{n \times n}$ where $a_{ij} = 0 \ \forall \ i<j$

Identity Matrix –

$$A = I_n = [a_{ij}]_{n\times n} \quad \text{where} \quad a_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Result :– $A_{n\times n} I_n = A_{n\times n}$

Invertible (Non-singular) – A square matrix $A_{n\times n}$ is invertible if there exist square matrix $B_{n\times n}$ such that

$\boxed{A \cdot B = B \cdot A = I_n}$ We write $\boxed{B = A^{-1}}$

Otherwise then A is non-invertible (singular).

Results :–

1) $(A^{-1})^{-1} = A$ 2) $(KA)^{-1} = \frac{1}{K} A^{-1}$ 3) $(AB)^{-1} = B^{-1}A^{-1}$

4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ 5) $I^{-1} = I$

Ex :– Let A, B, C, D, E, F are invertible matrices such that

$ABCDEF = I$, $D^{-1} =$ ______

a) ABCEF b) EFCBA [✓ c)] EFABC d) ABCFE

$$ABCDEF = I$$
$$D^{-1} = EFABC$$

Ex :– $ABC = I$, $B^{-1} =$ ____

$(ABC)^{-1} = I^{-1}$

$C^{-1}B^{-1}A^{-1} = I$

$C^{-1}B^{-1}A^{-1}A = IA \Rightarrow C^{-1}B^{-1} = IA$

$CC^{-1}B^{-1} = IA$

$B^{-1} = CIA$ or CA Ans.

Ex :– X & Y are singular matrices such that

$XY = Y$ & $YX = X$

$X^2 + Y^2 =$ ______

$$= YX + XY = X + Y$$

$\longrightarrow$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ & $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ (1) $X \cdot Y = x_1 y_1 + x_2 y_2 + \text{---} \, x_n y_n$.

(2) If $X \cdot Y = 0$ then X & Y are orthogonal

Ex :- $X = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $Y = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ $X \cdot Y = 2 - 2 = 0$ → orthogonal.

(3) Length or norm of a vector :

$$||X|| = \sqrt{x_1^2 + x_2^2 + \text{------} \, x_n^2}$$

Ex :- $X = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $||X|| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$

(4) Normalising vector : $\dfrac{X}{||X||}$ → Making new vector whose length is 1.

Ex :- $X = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ $||X|| = \sqrt{4 + 1^2 + 4} = \sqrt{9} = 3$

$$Y = \frac{X}{||X||} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$||Y|| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1$$

Orthonormal set – Let $X_1, X_2 \text{----} X_n$ be n vectors of dimension $n \times 1$. The set of vectors $X_1, X_2 \text{-----} X_n$ are said to be orthonormal.

$$X_i \cdot X_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Ex :- $X_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$, $X_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$X_1, X_2 \longrightarrow$ Orthonormal set.

Def – A square matrix A is

(1) Symmetric if $\boxed{A^T = A}$ (2) Skew symmetric if $\boxed{A^T = -A}$

(3) Orthogonal if $\boxed{A^T = A^{-1}}$ (or) $\boxed{A \cdot A^T = A^T \cdot A = I}$

Ex :- 1) $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ is symmetric matrix.

2) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is skew symmetric matrix

3) $A = \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $A \cdot A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

A is orthogonal matrix. $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

Q:- $A = \begin{bmatrix} 1 & 4 & 5 \\ a & 2 & 2 \\ b & c & 3 \end{bmatrix}$ is symm. then a, b, c = ?

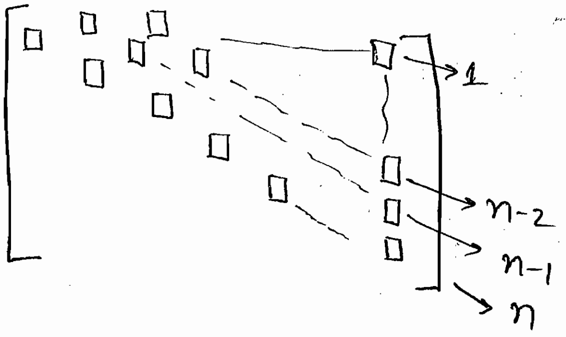
$a = 4, \; b = 5, \; c = 2.$

Q:* No. of symm. matrices of order n whose elements are either 0 or 1 = ______

if elements 0, 1, 2 then $3^{\text{Pow.}}$

a) 2^{n^2} b) $2^{\frac{n^2-n}{2}}$ ✓c) $\boxed{2^{\frac{n^2+n}{2}}}$ d) 2^n

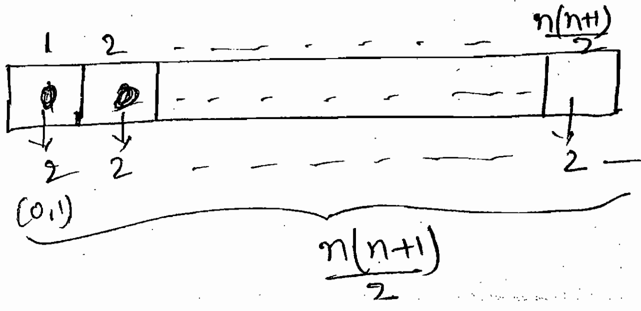
A =



To form a symmetric matrix of order n we need to fix (fill)

$= n + (n-1) + \text{-----} \; 2 + 1$

$= \frac{n(n+1)}{2}$ elements



$2^{\frac{n(n+1)}{2}} = 2^{\frac{n^2+n}{2}}$

$\longrightarrow$ Each position can be filled in 2 ways (0 or 1) be 0.

(2) If A is square matrix :

1. $A + A^T$ is Symmetric Matrix
2. $A - A^T$ is Skew Symm. Matrix

(1) $(A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

(2) $(A-A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$

3. $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

Every matrix can be expressed as sum of symmetric & skew symmetric matrix.

***4. The rows/columns of orthogonal matrix form orthonormal set

ex :- $A = \begin{bmatrix} R_1 \\ R_2 \\ R_n \end{bmatrix}_{n\times n} \rightarrow$ Row 1 is orthogonal.

then $R_i \cdot R_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Q:- $A = \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1/9 & -4/9 & 8/9 \\ 8/9 & 4/9 & 1/9 \\ \alpha/9 & -7/9 & \beta/9 \end{bmatrix}$ is orthogonal $\alpha = ?, \beta = ?$

Soln :- A is orthogonal $\Rightarrow R_1 \cdot R_3 = R_2 \cdot R_3 = 0$

$R_1 \cdot R_3 = 0 \Rightarrow \frac{\alpha + 28 + 8\beta}{81} = 0 \Rightarrow \alpha + 28 + 8\beta = 0$

$R_2 \cdot R_3 = 0 \Rightarrow \frac{8\alpha - 28 + \beta}{81} = 0 \Rightarrow 8\alpha - 28 + \beta = 0$

$$\begin{array}{cccc} & \alpha & \beta & 1 & \\ 8 & 28 & 1 & 8 \\ 1 & -28 & 8 & 1 \end{array}$$

$\frac{\alpha}{-28\times 8 - 28} = \frac{\beta}{28\times 28 + 28} = \frac{1}{1 - 8\times 8} = \frac{1}{-63}$

$\Rightarrow \alpha = 4$ & $\beta = -4$

L-5

Determinant – No. associated with square matrix

2×2 Matrix – $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

Ex:– $\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 2-6 = -4$

Minor of a_{ij} – Determinant of square submatrix obtained by deletin i^{th} row & j^{th} column.

Ex:– $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}$ $M_{31} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5$ $M_{23} = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = -4$

Cofactor of a_{ij} – $C_{ij} = (-1)^{i+j} M_{ij}$

$C_{31} = (-1)^{3+1} M_{31} = + M_{31} = +5$; $C_{22} = +8$; $C_{12} = -16$

Cofactor matrix of A = Cofactor A = $\begin{bmatrix} +3 & -16 & +2 \\ -14 & +8 & -(-4) \\ +5 & -0 & +(-10) \end{bmatrix}$

Laplace Expansion :– <u>nxn matrix</u>

<u>Row Expansion</u> – $a_{i1}C_{i1} + a_{i2}C_{i2} + \text{--------} a_{in}C_{in}$

<u>Column Expansion</u> – $a_{1j} C_{1j} + a_{2j}C_{2j} + \text{-------} a_{nj}C_{nj}$

Ex:– $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix} \Rightarrow |A| = 2(3) + 3(-16) + 1(2) = -40$

<u>Results</u> :–

1) Expansion of determinant of 2x2 matrix contain <u>2 terms</u>.
2) —"— —"— —"— of 3X3 —"— —"— <u>3·2 = 3! terms</u>.
3) —"— of nxn —"— <u>n! terms</u>.

(5) The matrix A has

(i) Zero row | Zero column

(ii) Two proportional rows/columns

then $|A| = 0$.

Ex :- $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 5 & 100 \end{bmatrix} \Rightarrow |A| = 0$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 5 & 2 \end{bmatrix}$ Proportional rows $R_2 = 2R_1$ $|A| = 0$

Ex :- $\begin{bmatrix} 4 & 5 & 6 \\ 0 & 7 & 1 \\ 0 & 0 & 5 \end{bmatrix} = |A| = 4 * 7 * 5 = 140$

(6) If A is diagonal or upper triangular or lower triangular matrix, then $|A|$ = Product of main diagonal entries.

$* \ |I| = 1$

(7) $A \xrightarrow{R_i \leftrightarrow R_j} B$, $|B| = -|A|$

(8) $A \xrightarrow[K \neq 0]{R_i \rightarrow KR_i} B$, $|B| = K|A|$

(9) $A \xrightarrow{R_i \rightarrow R_i + KR_j} B$, $|B| = |A|$

Ex :- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $|A| = 3$ $B = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$, $|B| = -3$

$C = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ 3g & 3h & 3i \end{bmatrix} = 2 * 3 * \underset{|A|}{\underset{\downarrow}{3}} = 18$

Ex :- $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & -5 & -2 \\ -6 & 2 & 3 \end{bmatrix}$ $R_1 \rightarrow R_1 + R_2 + R_3$, $= \begin{bmatrix} 0 & 0 & 0 \\ 4 & -5 & -2 \\ -6 & 2 & 3 \end{bmatrix}$

$|A| = 0$

(10) If elements of each row add upto zero, $|A| = 0$

(11) If A is $n \times n$ matrix, $|KA| = K^n |A|$

Ex :- $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} \Rightarrow |A| = 0 \rightarrow$ odd order (3×3)

(12) Determinant of skew symmetric matrix of odd order is 0.

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow |A| = 1 \rightarrow$ even-order (2×2)

(13) $|AB| = |A|\,|B|$

Adjoint of A - $\text{Adj } A = [\text{Cofactor } A]^T$

Formula :- $A^{-1} = \dfrac{\text{Adj } A}{|A|}$, $|A| \neq 0$

(1) ~~If~~ A is non singular iff $|A| \neq 0$

(2) A is singular iff $|A| = 0$

(3) If A is invertible, $|A^{-1}| = \dfrac{1}{|A|}$

Ex :- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times3}$, $|A| = 3$

Det $|(2A^{-1})| =$

$$|(2A)^{-1}| = \left|\frac{1}{2}A^{-1}\right| = \left(\frac{1}{2}\right)^3 \frac{1}{|A|} = \frac{1}{24}$$

Ex :- $A = \begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$; A is singular then $x =$

$|A| = 0 = -2(48 - 12x) \Rightarrow x = 4$ Ans.

(4) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(5) $\text{Adj } A = |A| A^{-1}$ (6) $(\text{Adj}A)^{-1} = \dfrac{A}{|A|}$

$$(\text{Adj}A)^{-1} = (|A|A^{-1})^{-1} = \frac{1}{|A|}(A^{-1})^{-1} = \frac{A}{|A|}$$

$\text{Adj } A = |A| A^{-1}$

$\therefore |\text{Adj } A| = \left| |A| A^{-1} \right| = |A|^n |A^{-1}| = \frac{|A|^n}{|A|} = |A|^{n-1}$

8. $\text{Adj}(\text{Adj } A) =$, $A_{n \times n}$

$\text{Adj } A = |A| A^{-1}$

$\text{Adj}(\text{Adj } A) = |\text{Adj } A| (\text{Adj } A)^{-1} = |A|^{n-1} \cdot \frac{A}{|A|} = |A|^{n-2} A$

Ex :- A is 3×3 matrix , $|A| = 4$

$|\text{Adj } A| = |A|^{3-1} = 4^2 = 16$

$\text{Adj}(\text{Adj } A) = |A|^{3-2} A = 4^1 A = 4A$

Q:- $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$ $\qquad \text{Adj} A = \begin{bmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & K & 7 \end{bmatrix}$

$K = ?$, $A^{-1} = ?$

$\text{Adjoint } A = [\text{Cofactor } A]^T$

$k = a_{32}$ of adj A $=$ Cofactor a_{23} of $A = (-1)^{2+3} \begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix}$

$= -5$

$A^{-1} = \frac{\text{Adj } A}{|A|}$

$|A| = -11 \times 1 + (-9) \times 2 + 1 \times 0 = -11 - 18 = -29$

Q:- $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$ Top row of $R^{-1} = ?$

a) $[5 \; 6 \; 7]$ [✓ b)] $[5 \; -3 \; 1]$ c) $[2 \; 0 \; -1]$ d) $[2 \; -1 \; 0]$

$$|R| \qquad |R|$$

$$1^{st} \text{ row of } R^{-1} = \frac{\text{First row of Adj } R}{|R|} = \frac{[\text{First column cofactor } R]^T}{|R|}$$

$$= \frac{\begin{bmatrix} 5 & -3 & +1 \end{bmatrix}}{|R|}$$

$$|R| = 1(5) + 2(-3) + 2(1) = 5 - 6 + 2 = 1$$

Find 2^{nd} row of R^{-1} & 3^{rd} row of R^{-1}

Rank –

Minor – Determinant of any square submatrix.

Ex :– $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 1 & 2 \end{bmatrix}$

(I) 1×1 Submatrices → $[1] \quad [2]$ ------

Minors of order 1 → $1 \quad 2$

(II) 2×2 Submatrices → $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$

Minors of order 2 → $\Downarrow 1 \quad \Downarrow 1 \quad 0$

(III) 3×3 Submatrices → $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}$ ------

Minors of order 3 → $\Downarrow -5 \quad \Downarrow 2$ $\qquad \rho(A) = 3$

Rank :– The order of highest order non-Zero minor.
(vanishing)

Method :– The no. of non-Zero rows in echelon form is rank of A.

Results –

1) A matrix A is said to be of rank r if

of order r.

1(2) every minor of order higher than r (if exist) must be zero.

Ex :- $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad \rho(A) = 2$

2) $\rho(A) = 0$ iff $A = 0$ (Zero matrix)

3) $\rho(A^T) = \rho(A)$

4) $\rho(A_{m \times n}) \leq \text{Min}\{m, n\}$

5) $\rho(A+B) \leq \rho(A) + \rho(B)$

Ex :- $A_{3\times4}$ $\quad B_{3\times4}$ $\quad (A+B)_{3\times4}$

$$\text{Max}^m \text{Rank} \quad \overset{A_{3\times4}}{\downarrow}\, 3 + \overset{B_{3\times4}}{\downarrow}\, 3 \geq \overset{(A+B)_{3\times4}}{\downarrow}\, 3$$

6) $\rho(A-B) \geq \rho(A) - \rho(B)$

Ex :- $A_{3\times4}$ $\quad B_{3\times4}$ $\quad (A-B)_{3\times4}$

$$\text{Max}^m \text{Rank} \quad \downarrow 3 - \downarrow 3 \leq \downarrow 3$$

* 7) $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

Ex :- $A_{3\times4}$ $\quad B_{4\times6}$ $\quad (AB)_{3\times6}$

$$\text{Max}^m \text{Rank} \quad \downarrow 3 \quad \downarrow 4 \quad \downarrow 3$$

8) Let $A_{n\times1}$ & $B_{1\times n}$ non zero matrices

$$\boxed{\rho(AB) = 1}$$

Ex :- $A = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}_{3\times1}$ $\quad B = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}_{1\times3}$ $\quad AB = \begin{bmatrix} 4 & 2 & 2 \\ 8 & 4 & 4 \\ 6 & 3 & 3 \end{bmatrix}_{3\times3}$

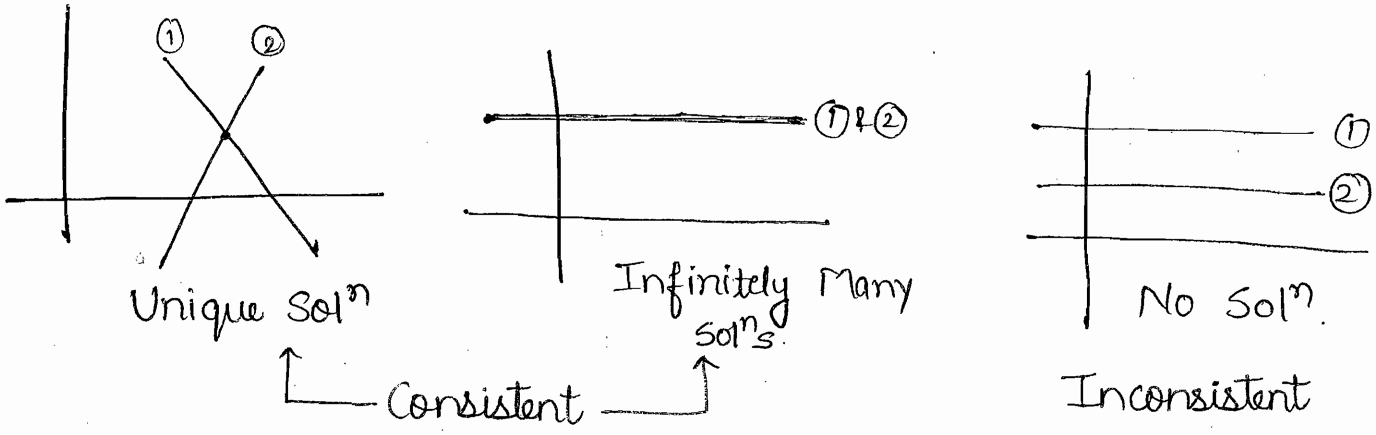
$$= \begin{bmatrix} 2R_1 \\ 4R_1 \\ 3R_1 \end{bmatrix} \text{ Proportional rows}$$

$$= \begin{bmatrix} 4 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$$

$\hookrightarrow$ Rank = 1

Ex :— $a_1x + b_1y = c_1$ — ①

$a_2x + b_2y = c_2$ — ②



System of m linear equations in n unknown :

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{m2}x_2 + \cdots\cdots a_{mn}x_n = b_m \end{array}$$

(i) $m < n$ under determined System.

(ii) $m = n$ Determined system.

(iii) $m > n$ Over determined System.

Matrix form :

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

$$\boxed{\underset{m\times n}{A}\,\underset{n\times 1}{X} = b_{m\times 1}}$$

A → Coefficient Matrix

X → Soln Vector

b → Column Vector

$b \equiv 0$, Homogeneous Linear System (HLS)

$b \neq 0$, Non —"— —"— —"— (NHLS)

Augmented Matrix : $[A \vdots B]$.

$$A_{m\times n} X_{n\times 1} = b_{m\times 1}$$

$\downarrow$ n = No. of unknowns

$\rho(A:b) \neq \rho(A)$
No Sol^n
(Inconsistent)

$\rho(A) = \rho(A:b) = r$
Consistent

Unique Sol^n
$\boxed{r = n}$

Infinite Sol^n
$\boxed{r < n}$

In case of infinitely many sol^n :—

No. of Linearly independent solutions = No. of free variables
$= n - r$.

Ex :— $x + y + z = 6$
$x + 2y + 3z = 4$
$x + 2y + 3z = 5$

Matrix form : $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$

$A \quad X \quad = b$

Augmented Matrix : $\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \end{array}\right]$

$R_3 \to R_3 - R_2$

$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 - R_1$

$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Q:- $x+y+z=6$; $x+2y+3z=4$; $x+2y+4z=5$

Sol :- Augmented matrix :

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 5 \end{array}\right]$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Echelon form

$\rho(A) = \rho(A:B) = 3 = r \to$ Consistent

$r = n$

$\therefore$ Unique Solⁿ.

Method 1 - Back elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & \boxed{1} & 6 \\ 0 & 1 & \boxed{2} & -2 \\ 0 & 0 & \textcircled{1} & 1 \end{array}\right]$$

$R_1 \to R_1 - R_3$ & $R_2 \to R_2 - 2R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Reduced echelon form

$x = 9$
$y = -4$
$z = 1$

$$X = \begin{bmatrix} 9 \\ -4 \\ 1 \end{bmatrix}$$

from reduced echelon form

Method 2 - (Back Substitution) [For unique solⁿ].

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Echelon Form

$x+y+z=6$
$y+2z=-2$
$\boxed{z=1}$

$y+2(1)=-2$
$\boxed{y=-4}$
$x+(-4)+1 = 6 \Rightarrow \boxed{x=9}$

Imp

Q:- $x+y+z=6$
$x+2y+3z=4$
$x+2y+3z=4$

$$(A:B) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Echelon Form

$\rho(A) = \rho(A \vdots B) \Rightarrow 2 = r \rightarrow$ Consistent

$n = 3 \quad \boxed{r < n}$

Infinitely Many solution

No. of L.I. solutions $= n - r = 3 - 2 = 1$

Back elimination :

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 8 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced row echelon form.

Basic Variables (Dependent variables) –

Variables corresponding leading 1's in reduced row echelon form

$\underset{\text{Basic variables}}{B.V} \longrightarrow x, y.$

other variables are free variables

$\underset{\text{free variables}}{f.V} \rightarrow z$

Let $\boxed{z = t}$ $\quad x - z = 8, \quad y + 2z = -2$

$x = 8 + t, \quad y = -2 - 2t, \quad z = t$

$$X = \begin{bmatrix} 8+t \\ -2-2t \\ t \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 0 \end{bmatrix} + t \underset{\substack{\downarrow \\ \text{1 L.I. sol}^n \text{ which create all} \\ \text{Infinite solutions.}}}{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

Ex:– $x + y + z = 60$
$x + 4y + 6z = 20$
$x + 4y + \lambda z = \mu$

For what values of λ & μ, system has no sol^n, unique sol^n, infinite solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 60 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 60 \\ 0 & 3 & 5 & -40 \\ 0 & 3 & \lambda-1 & \mu-60 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 60 \\ 0 & 3 & 5 & -40 \\ 0 & 0 & \lambda-6 & \mu-20 \end{array}\right]$$

Echelon form

(1) No solⁿ

$\lambda - 6 = 0 \ , \ \mu - 20 \neq 0$

$\lambda = 6 \ , \ \mu \neq 20$

$\rho(A) = 2 \ , \ \rho(A:B) = 3$

(2) Infinite solⁿ

$\lambda - 6 = 0 \ , \ \mu - 20 = 0$

$\lambda = 6 \ \& \ \mu = 20$

$\rho(A) = \rho(A:B) = 2 = \gamma$

$\gamma < n(3)$

$2 < 3$

(3) Unique solⁿ

$\lambda \neq 6$ & μ can be any value

$\rho(A) = \rho(A:b) = 3 = \gamma = n$

Homogeneous Linear System – $AX = 0$

(1) $X = 0$ is trivial solⁿ

$\therefore AX = 0$ is always consistent.

$$A_{m\times n} X_{n\times 1} = 0$$

↓ No. of Unknowns

$\rho(A) = n$	$\rho(A) < n$
Unique solⁿ	Infinite solⁿ (or)
Zero solⁿ	Non-zero solⁿ
$\boxed{X = 0}$	(or) $\boxed{X \neq 0}$

Results –

(I) $AX = 0$ & A is square matrix of order n.

1. Unique solⁿ iff $\rho(A) = n$; iff $|A| \neq 0$.

(II.) An underdetermined HLS always has infinite solution.

$A_{m\times n} X_{n\times 1} = 0$; $n \rightarrow$ no. of unknowns

Underdetermined $m<n$; $\rho(A) \le \min\{m,n\}$

$\boxed{\rho(A) \le n}$ Infinite sol^n.

Ex :- $x+y+z=0$, $x+2y+3z=0$, $x+2y+4z=0$

a) Unique Sol^n b) Infinite sol^n c) Exactly two sol^n d) None

Sol^n :- Method 1 :

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

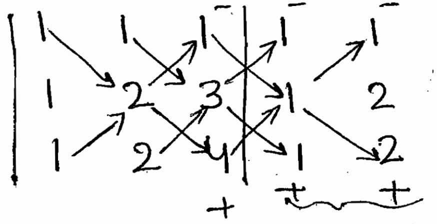
$\rho(A) = 3 = n \rightarrow$ Unique Sol^n , $\boxed{X=0}$

Method 2 :

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

Shortcut for det. of 3X3 matrix only

|A| =



repeat two columns

$= 8+3+2-2-6-4 = 1$

$|A| \neq 0 \Rightarrow \rho(A) = 3$

Unique Sol^n.

Q:- $x+y+z=0$, $x+2y+3z=0$

$\rightarrow$ It is underdetermined HLS , Rank = 2

$\rho(A) = 2 < n \Rightarrow$ infinite sol^n.

$x + 3y + 4z = 0$

$x + 3y + 4z = 0$

Solⁿ :–
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 3 & 4 \end{bmatrix} X = 0$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Echelon form
$\rho(A) = 2 < 3$
$\Rightarrow$ Infinite Sol's.

$R_2 \rightarrow R_2/2$

$R_1 \rightarrow R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon
B.V $\rightarrow x, y$
f.V $\rightarrow z$

$z = 2t$

$x - \frac{1}{2}z = 0$

$y + \frac{3z}{2} = 0$

$x = t$

$y = -3t$

$z = 2t$

$$X = \begin{bmatrix} t \\ -3t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

Result :– In case HLS with infinite solutions, No. of. L.I. sol's $\boxed{= \text{Nullity} = n - r}$

Ex :– $x + y + z = 0$, Nullity = ____

Solⁿ :– $r = 1$, $n = 3$, Nullity $= 3 - 1 = 2$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} X = 0$$

$\uparrow$
Echelon form, reduced row echelon, $\rho(A) = r = 1$

Ex :– The system $ax + y + z = 0$
$x + ay + z = 0$
$x + y + az = 0$ has infinite solⁿ.

The set of values of a for this to happen is

a) $\{1, -1\}$ b) $\{-1, 2\}$ c) $\{-2, 2\}$ d) $\{-2, 1\}$

$(A) < 3 \Rightarrow |A| = 0$

$$A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}_{3\times3}$$

$R_1 \to R_1 + R_2 + R_3$

$$(a+2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix}_{3\times3}$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$(a+2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & a-1 & 0 \\ 0 & a & a-1 \end{vmatrix} = 0$$

$$(a+2)(a-1)^2 = 0 \Rightarrow \boxed{a = -2, 1, 1} \Rightarrow \{1, -2\}$$

Linearly Independent vectors –

Ex :– $V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $V_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $V_2 = 2 \cdot V_1 \to$ Linearly Dependent

Ex :– $V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ V_1, V_2 cannot be expressed one in terms of other,

$V_1, V_2 \to$ L.I.

Result :– Let $V_1, V_2, \ldots V_n$ be n vectors & let $A = [V_1\ V_2 \ldots V_n]$

(i) $\rho(A) = n \Rightarrow V_1, V_2 \ldots V_n$ are L.I..

(ii) $\rho(A) < n \Rightarrow V_1, V_2 \ldots V_n$ are L.D..

Ex :– $A = [V_1\ V_2] = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{2\times2} \Rightarrow |A| = 4 - 4 = 0 \to$ L.D.

$\because \rho(A) < 2$

Ex :– $A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \Rightarrow |A| = -2 - 2 = -4 \neq 0 \to V_1, V_2$ are L.I..

Ex :– $V_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $V_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \rho(A) < 3 \to \text{L.D.}$$

Def :- Let 'A' be square matrix of order n.
A scalar "λ" is eigen value of A
if there exist a non-zero vector X $(X \neq 0)$
such that $\boxed{AX = \lambda X}$

X is eigen vector for λ

Characteristic eqⁿ - $AX = \lambda X \Rightarrow AX = \lambda I X \Rightarrow AX - \lambda I X = 0$

$\boxed{(A-\lambda I)_{n\times n} X = 0} \to$ H.L.S.

$X \neq 0 \Rightarrow$ Infinite solⁿ. $\Rightarrow \boxed{|A-\lambda I| = 0} \to$ C.E.

Method :- Let $A_{n\times n}$ be square matrix

1) Solve $|A-\lambda I| = 0$ for eigen values $\lambda = \lambda_1, \lambda_2 \ldots$
2) For each eigen value λ. Solve the HLS $(A-\lambda I)X = 0$ for eigen vectors $X \neq 0$

Q:- $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ Find eigen values & eigen vectors

Solⁿ :- 1) Solve $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda) - 4 = 0$$

$$\cancel{4} - 5\lambda + \lambda^2 - \cancel{4} = 0$$

$$\lambda^2 - 5\lambda = 0$$

$$\lambda(\lambda - 5) = 0$$

$$\lambda = 0, 5 \to \text{eigen values}$$

$\lambda_1 = 0$ & $\lambda_2 = 5$

2) Eigen vector for $\lambda = 0$:

$$(A-\lambda I)X = 0$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \text{ reduced row echelon}$$

B.V → x , f.V $= y \Rightarrow y = t$, $x + 2y = 0$

$x = -2t$

$$X = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

For $\lambda_1 = 0$, $X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a eigen vector

↳ simple integer form, can be $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -20 \\ 10 \end{bmatrix}$ etc.

$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} X = 0$$

$R_2 \to 2R_2 + R_1$ $\qquad$ $R_1 \to -R_1/4$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} X = 0 \Rightarrow A \sim \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

Echelon form $\qquad$ Reduced row echelon form

B.V $\to x$, f.V. $\to y$

$y = 2t$, $x - \frac{1}{2}y = 0 \Rightarrow x = t \Rightarrow y = 2t$

$$X = \begin{bmatrix} t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda_2 = 5$, $X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a eigen vector.

Trace :— $A_{n \times n}$

$\text{tr}A$ = Sum of main diagonal entries = Sum of eigen values

Ex:— $\text{tr}\,A = 1 + 4 = 5 = \lambda_1 + \lambda_2$

$|A| = 0 = \lambda_1 \cdot \lambda_2 \longrightarrow$ Det. = Product of eigen values.

Q:— $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Sol^n:— 1) C.E. $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$$

$\lambda^3 = 0$

$\boxed{\lambda = 0} \to$ "0" is eigen value of algebraic multiplicity 3.

Eigen vector for $\lambda = 0$

$[A - 0I]X = 0$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X = 0 \qquad \begin{bmatrix} 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \text{B.V.} \to \text{leading element.}$$

Reduced echelon form.

$\boxed{x = t \qquad y = s}$ $\qquad z = 0$

~~x~~ $$X = \begin{bmatrix} t \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are eigen vectors for $\boxed{\lambda = 0}$

→ no. of eigen vectors

"0" is eigen value with geometric multiplicity 2.

Note – Geometric Multiplicity ≤ Algebraic Multiplicity.

Q:– $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ then eigen value for eigen vector $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ = ____

a) 2 b) 4 c) 6 d) 1

Solⁿ:— $AX = \lambda X$

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 101 \\ 101 \end{bmatrix} = \begin{bmatrix} 606 \\ 606 \end{bmatrix} = \underset{\text{Ans}}{\textcircled{6}} \begin{bmatrix} 101 \\ 101 \end{bmatrix}$$

Q:– A matrix M has eigen values $\underset{\lambda_1}{1}$ & $\underset{\lambda_2}{4}$ with eigen vector $\underset{X_1}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$ & $\underset{X_2}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$ resp. $M =$ ____

Solⁿ:— $AX = \lambda X \Rightarrow MX = \lambda X$

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$

$MX_1 = \lambda_1 X_1 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$a - b = 1$, $c - d = -1$

$MX_2 = \lambda_2 X_2 \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} 2a + b = 8 \\ 2c + d = 4 \end{matrix}$

$a = 3$, $b = 2$, $c = 1$, $d = 2$

$M = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ Ans.

(1) $\text{Tr } A$ = Sum of eigen values.

(2) $|A|$ = Product eigen values.

(3) $|A| = 0$ iff $\lambda = 0$ is an eigen value.

(4) Eigen vectors corresponding distinct eigen values are L.I.

Ex:— $M = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ $\quad \lambda_1 = 1, \; X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad A = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$

$\lambda_2 = 4, \; X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$|A| = 3 \neq 0 \rightarrow X_1, X_2$ are L.I.

(5) Eigen vectors corresponding to distinct eigen values of symm. matrix are orthogonal.

Ex:— $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow$ Symm. Matrix

$\lambda = 0 \quad X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \lambda = \lambda, \quad X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$X_1 \cdot X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0 \Rightarrow X_1, X_2$ are orthogonal.

Q:— $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \lambda_1 = ? \;, \lambda_2 = ?$

Sol^n:— $\lambda_1 + \lambda_2 = \text{tr} A = 1 + 1 = 2$

$\lambda_1 \cdot \lambda_2 = |A| = 0$

Let $\lambda_1 = 0, \lambda_2 = 2 \Rightarrow$ Eigen values 0,2.

(6) Eigen values of matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times2}$ are 2, 0

———— || ———— $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times3}$ are 3, 0, 0

———— || ———— $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4\times4}$ are 4, 0, 0, 0

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{n\times n} = n, \underbrace{0, 0, \cdots 0}_{n-1 \text{ times}}$$

(7) If λ is eigen value of A then

(i) eigen value of KA is $K\cdot\lambda$.

(ii) eigen value of $A+KI$ is $\lambda \pm K$.

(iii) —"— —"— of A^2 is λ^2.

(iv) —"— —"— of A^m is λ^m.

(8) If λ is eigen of an invertible matrix A

(i) Eigen value of $A^{-1} = 1/\lambda$

(ii) —"— —"— of $\text{Adj}\,A = \frac{|A|}{\lambda}$

Q:— Let $1, -2, -1$ are eigen values of $A_{3\times3}$.

(1) Then eigen values of $3A = 3, -6, -3$

(2) ——"—— $A+3I = 1+3, -2+3, -1+3 = 4, 1, 2$

(3) ——"—— $A^2 = 1^2, -2^2, -1^2 = 1, 4, 1$

(4) ——"—— $A^5 = 1^5, -2^5, -1^5 = 1, -32, -1$

(5) ——"—— $A^{-1} = 1/1, 1/-2, 1/-1 = 1, -1/2, -1$

(6) ——"—— $\text{Adj}\,A = 2/1, 2/-2, 2/-1 = 2, -1, -2$

$|A| = 1 \times -2 \times -1 = 2$

(7) ——"—— $A-4I = 1-4, -2-4, -2-1 = -3, -6, -5$.

(8) $|A^2+4A| = |A(A+4I)| = |A|\,|A+4I|$

$|A| = 1 \times -2, -1 = 2$

$|A+4I| = 5 \times 2 \times 3 = 30$

$|A^2+4A| = 2 \times 30 = 60$

Ex:— Let $1, -1, 0$ are eigen value then $|A^{100}+I| =$ ____

Soln:— Eigen values of $A^{100} = 1^{100}, -1^{100}, 0^{100} = 1, 1, 0$

——"—— $A^{100}+I = 1+1, 1+1, 0+1 = 2, 2, 1$

$|A^{100}+I| = 2 \times 2 \times 1 = 4$ Ans.

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 1 & 4 \end{bmatrix} \rightarrow \text{lower } \Delta\text{lar}$$

Eigen values of $A = 2, 1, 4$.

(9) If A is diagonal matrix or upper triangular or lower triangular matrix, then eigen values are the main diagonal entries.

Q:— $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ $\qquad |A| = \_\_\_\_$

Soln:— $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A = B + I$

$|A| = |B+I|$
$= 5 * 1 * 1 * 1$
$= 5$

Eigen values of $B \rightarrow 4, 0, 0, 0$

$B+I \rightarrow 5, 1, 1, 1$

Complex Matrices — Let $A = \begin{bmatrix} 1+i & 2+3i \\ 4-i & 5 \end{bmatrix}$

Conjugate of $A : \bar{A}$

$\bar{A} = \begin{bmatrix} 1-i & 2-3i \\ 4+i & 5 \end{bmatrix}$

Conjugate transpose — $\bar{A}^T = A^\theta$.

$\bar{A}^T = \begin{bmatrix} 1-i & 4+i \\ 2-3i & 5 \end{bmatrix}$

Note :— A is real matrix $\boxed{\bar{A}^T = A^T}$

Def :— A complex square matrix A is :

1) Hermitian — if $\boxed{\bar{A}^T = A}$

2) Skew Hermitian — if $\boxed{\bar{A}^T = -A}$

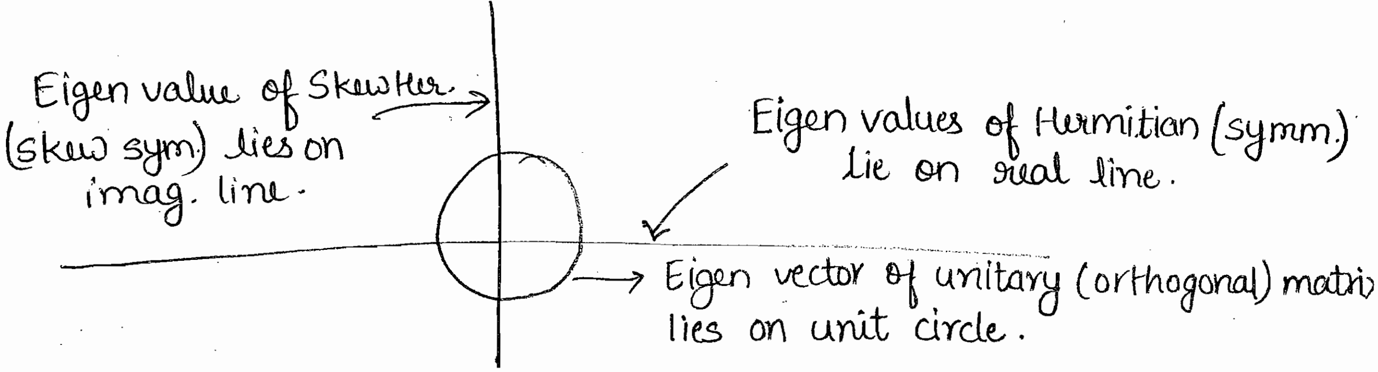
Ex :- $A = \begin{bmatrix} 2 & 2-3i \\ 2+3i & 3 \end{bmatrix}$ is Hermitian

Ex :- $A = \begin{bmatrix} 2i & 2+3i \\ -2+3i & 0 \end{bmatrix}$ $\bar{A}^T = \begin{bmatrix} -2i & -2-3i \\ 2-3i & 0 \end{bmatrix} = -\begin{bmatrix} 2i & 2+3i \\ -2+3i & 0 \end{bmatrix}$

Ex :- $A = \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\bar{A}^T = \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$A\bar{A}^T = I \rightarrow A$ is unitary

*



(10) Eigen values of Hermitian (or) Symm. are real no.

Ex :- $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ e.v. → 0, 5 → real no.

(11) Eigen values of Skew-Hermitian or skew-symm. matrix are either pure imaginery or "0".

Ex :- $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\lambda^2 + 1 = 0$

$\lambda = \pm i \leftarrow$ pure imag.

(12) Eigen values of unitary (orthogonal) matrix have absolute value 1.

Ex :- $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ Orthogonal Eigen value $= \frac{1 \pm i}{\sqrt{2}}$

$\lambda_1 = \frac{1+i}{\sqrt{2}}$, $|\lambda_1| = 1$; $\lambda_2 = \frac{1-i}{\sqrt{2}} \Rightarrow |\lambda_2| = 1$

Every square matrix satisfies its C.E.

Ex:— $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ C.E. $\rightarrow$ $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\boxed{\lambda^2 - 3\lambda - 4 = 0} \longrightarrow \text{Characteristic Polynomial eq}^n$$

$|A| \rightarrow$ const. term in

By Cayley Hamilton Theorem:

$$\boxed{A^2 - 3A - 4I = 0}$$

Finding A^{-1}:— Cayley Hamilton theorem is used to find A^{-1}.

$$A^2 - 3A - 4I = 0$$

Multiply A^{-1} $\longrightarrow$ $A - 3I - 4A^{-1} = 0$

$$A^{-1} = \frac{1}{4}\cancel{3I}(A - 3I)$$

$$= \frac{1}{4}\begin{bmatrix} -2 & 2 \\ 3 & -1 \end{bmatrix}$$

Finding powers of A — Cayley H. theorem is used to find A^m.

$$\boxed{A^2 = 3A + 4I}$$

Multiply by A :

$$A^3 = 3A^2 + 4A$$

$$= 3[3A + 4I] + 4A = \boxed{13A + 12I = A^3}$$

Multiply by A :

$$A^4 = 3A^3 + 4A^2 = 3[13A + 12I] + 4[3A + 4I]$$

$$A^6 = \cancel{B}\, A^3 A^3$$

$$= (13A + 12I)(13A + 12I)$$

$$= 169A^2 + 312A + 144I = 169(3A + 4I) + 312A + 144I$$

$$\boxed{A^6 = 819A + 820I}$$

then $B = A^4 - 5A^2 + 5I$, $|B| =$ ______

Solⁿ :— C.E.

$$(\lambda-1)(\lambda+1)(\lambda-2)(\lambda+2) = 0$$

$$(\lambda^2-1)(\lambda^2-4) = 0$$

$$\boxed{\lambda^4 - 5\lambda^2 + 4 = 0} \rightarrow \text{C.E.}$$

By Cayley H. theorem : $A^4 - 5A^2 + 4I = 0$

$B = A^4 - 5A^2 + 5I$

$B = A^4 - 5A^2 + 4I + I$ $\quad\quad$ $B = I$

$B = 0 + I$ $\quad\quad$ $|B| = 1$

PROBABILITY

Random experiment (RE) – An experiment in which outcome is not certain.

Sample Space – Set of all possible outcomes of a random experimen

Event – Any subset of sample space is called event.

Ex :— Let S be a set with n elements.

No. of subsets $= 2^n$.

Ex :—

R.E.	Sample Space	Event
1) Tossing two coins	$S = \{HH, HT, TH, TT\}$ $2^4 \rightarrow$ distinct problms $\rightarrow 16$ Events	E_1 = getting exactly one head $= \{HT, TH\}$ E_2 = getting no heads $= \{TT\}$ E_3 = getting Rajnikant $= \{\ \}$
2) Selecting an integer from 1 to 20 at random.	$S = \{1, 2 \ldots\ldots\ldots 20\}$ $2^{20} \rightarrow$ distinct problms.	E_1 = getting multiple of 3 $= \{3, 6, 9, 12, 15, 18\}$ $|E_1| = |20| = 6$

getting Multiple of 5

$|E_2| = \left[\frac{20}{5}\right] = 4$

E_3 = Multiple of 3 & 5

$|E_3| = \left[\frac{20}{3\times5}\right] = 1$

Def :– For an event A,

$$\text{Probability of } A = P(A) = \frac{\text{favourable no. of outcomes}}{\text{total no. of outcomes}} = \frac{|A|}{|S|}$$

$$= \frac{m}{n}$$

(1) The odds in favour of A = $\underset{\text{fav}}{m} : \underset{\text{non-fav.}}{n-m}$

(2) The odds against A = $\underset{\text{non-fav}}{n-m} : \underset{\text{fav}}{m}$

Ex :– odds in favour of A are $\overset{\text{fav}}{2} : \overset{\text{Non.- fav.}}{5}$ & odds against B are $\underset{\text{non fav}}{3} : \overset{\text{fav}}{4}$ $P(A) = ?$ $P(B) = ?$

Soln :– Total = 5+2 = 7 $P(A) = \frac{2}{7}$

Total = 3+4 = 7 $P(B) = \frac{4}{7}$

Axioms :– (Universally accepted)

(1) For any event A of Sample space S, $0 \le P(A) \le 1$

(2) $P(S) = 1$

(3) If A & B are mutually exclusive (disjoint) events

$$P(A \cup B) = P(A) + P(B)$$

Results :–

1. Prob. of impossible event, $P(\phi) = 0$
2. Prob. of non happening of an event, $P(A^c) = 1 - P(A)$

Ex :- (1) Two dice are rolled.

(i) P[Sum of outcomes = 8]

Sum = 8 $\longrightarrow$ (2,6) (3,5) (4,4) (5,3)(6,2) $\longrightarrow$ 5

Two dice rolled :

Sum	2	3	4	5	6	7	8	9	10	11	12
Fav. case	1	2	3	4	5	6	5	4	3	2	1

(i) $P[\text{Sum } 3 \text{ or } 5] = P[3] + P[5] = \frac{2+4}{36} = \frac{6}{36} = \frac{1}{6}$

(ii) $P[\text{Sum} = \text{Prime no.}] = P[3 \text{ or } 5 \text{ or } 7 \text{ or } 11]$

$= \frac{1+2+4+6+2}{36} = \frac{15}{36}$

(iii) $P[\overset{\text{Sum} =}{\text{even no.}}] = P[2 \text{ or } 4 \text{ or } 6 \text{ or } 8 \text{ or } 10 \text{ or } 12]$

$= \frac{1+3+5+5+3+1}{36} = \frac{18}{36} = \frac{1}{2}$

(iv) $P[\text{Sum} = \text{odd no}] = 1 - P[\text{even no.}] = 1 - \frac{1}{2} = \frac{1}{2}$

(v) $P[\text{atleast } 3 = \text{sum}] = 1 - P[\text{Sum} = \text{atmost } 2]$

$= 1 - P[S \leq 2]$

$= 1 - \frac{1}{36} = \frac{35}{36}$

(vi) $P[\text{unequal outcomes}] = 1 - P[\text{equal outcomes on both dice}]$

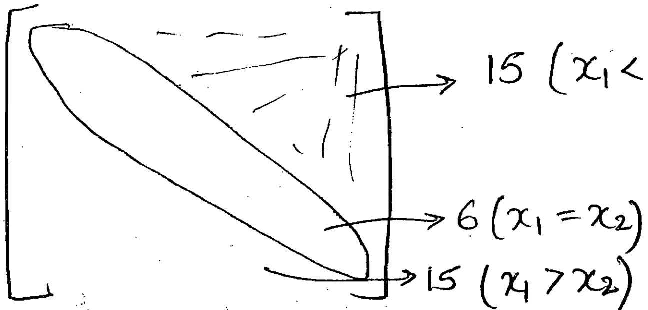
$= 1 - \frac{6}{36} = \frac{\cancel{30}^{5}}{\cancel{36}_{6}} = \frac{5}{6}$

every outcome has same chance

Q:- A fair dice is rolled twice.

P[outcome on the second toss is greater than the outcome in the first toss] = ?

Soln:-



$\longrightarrow P[\text{--}] = \frac{\cancel{15}^{5}}{\cancel{36}_{12}} = \frac{5}{12}$

30% families are having 2 children. & remaining families having 1C. A child is selected at random.

[i] P[child is from family having 2C] –

(ii) P[child is from family having 1C] –

Sol^n:– N → No. of families.

$$\text{Total no. of children} = \frac{50}{100} \times N * 3 + \frac{30}{100} \times N * 2 + \frac{20}{100} * N * 1$$

$$= \frac{23N}{10}$$

$$P[\text{child is from family having 2C}] = \frac{\text{fav.}}{\text{Total}} = \frac{6N/10}{23N/10} = \frac{6}{23}$$

$$P[\text{ from 1C}] = \frac{2N/10}{23N/10} = \frac{2}{23}$$

Q:– A coin is tossed N times.

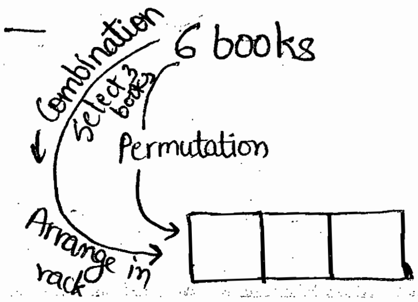
P[~~No. of heads~~ Diff. b/w No. of heads & no. of tails $= n-3$] $= \frac{0}{2^n} = 0$

Sol^n:–

No. of H	No. of T	Difference
n	0	n
$n-1$	1	$n-2$
$n-2$	2	$n-4$
⋮	⋮	
2	$n-2$	$4-n$
1	$n-1$	$2-n$
0	n	$-n$

fav = 0

Permutations & Combinations –



1) No. of r- permutations (Arrangements) of n-objects = $^{n}P_{r} = \frac{n!}{(n-r)!}$

2) No. of permutation (arrangements) of n-objects (in line) = n!

3) No. of circular permutations of n-objects = (n-1)!

[Fix one object & arrange remaining linearly]

Q:- How many ways 8 people can be arranged in a line?

8! ways

a) so that certain pair is always together. → 7! × 2!

$\boxed{AB}$

1 unit + 6 units = 7 units

$7\text{ units} \xrightarrow{\text{in line}} 7!$

$AB \longrightarrow 2!$

Q:- How many ways 8 people can arranged in a circle.

(8-1)! = 7!

a) so that certain pair is always together. 6! × 2!

$7\text{ units} \xrightarrow{\text{in circle}} 6!$

$AB \longrightarrow 2!$

Q:- 8 people arranged in a line.

(1) $P[\text{certain pair always together}] = P[\boxed{AB}] = \frac{AB}{8!} = \frac{2! \times 7!}{8!}$

$= \frac{2}{8} = \frac{1}{4}$

(2) $P[\text{certain pair never together}] = P[\boxed{\cancel{AB}}]$

$= 1 - \frac{1}{4} = \frac{3}{4}$

Q:- Above prblm for circle.

(1) $P[\boxed{AB}] = \frac{2! \times 6!}{7!} = \frac{2}{7}$

(2) $P[\cancel{AB}] = 1 - \frac{2}{7} = \frac{5}{7}$

Q:- 5 men & 5 women are arranged in a line.

1) P[all 5M together] 2) P[no two men together]

Sol^n:— Total Possibilities → 5M & 5W can be arranged $= (5+5)! = 10!$

1) all 5 men together → $5! \times 6!$

$\boxed{5M}$

1 unit + 5 unit = 6 units → 6! ways

5M → 5!

$$P[\text{all 5M together}] = \frac{5! \times 6!}{10!}$$

2) No two men together =

5W → 5! ways → No. constraint on W, consider first W.

$\times W_1 \times W_2 \times W_3 \times W_4 \times W_5 \times$

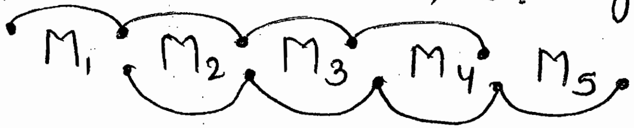
5M in 6 gaps → ${}^6P_5$ ways.

No two men together $= 5! \times {}^6P_5$

$$P[\text{No 2M together}] = \frac{5! \times {}^6P_5}{10!}$$

3) M & W arranged alternatively:—

5M → 5! ways



To get alternate arrangement 5W can be arranged (5! + 5!) ways

Men & women arranged alternately $= 5!(5! + 5!)$

$$P[\text{M \& W alternately}] = \frac{5!(5! + 5!)}{10!}$$

Q:— A box contains 3R & 4W balls. Balls are drawn from box one after the other at random & arranged in a line.

P[they are arranged alternatively] = ?

$$\frac{\text{R \& W alternately}}{7!} = \frac{4! \times 3!}{7!}$$

$4W \longrightarrow 4!$

$w_1 \times w_2 \times w_3 \times w_4 \longrightarrow 3R$ in $3!$ ways.

Combinations :—

1) No. of r-combinations (r selections) of n objects $= {}^nC_r = \frac{n!}{r!(n-r)!}$

2) ${}^nP_r = {}^nC_r * r!$

3) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \text{------} {}^nC_n = 2^n$

4) ${}^nC_0 + {}^nC_2 + {}^nC_4 + \text{------} = 2^{n-1}$

5) ${}^nC_1 + {}^nC_3 + {}^nC_5 + \text{------} = 2^{n-1}$

6) $0 * {}^nC_0 + 1 \times {}^nC_1 + 2 \times {}^nC_2 + \text{-----} n \times {}^nC_n = n \times 2^{n-1}$

Q:– 10 friends meet in a party & shake hands with each other

No. of hand shakes = No. of 2-selections $= {}^{10}C_2 = 45$

$A \leftrightarrow B$

2) Send greeting cards to each other → No. of cards send/posted

$= {}^{10}P_2 = 90$

$A \rightleftarrows B$

Q:– How many ways 5 members committee can be selected from 6M & 7W.

Sol^n:– ${}^{13}C_5$

2) With exactly 2 women in the committee $\rightarrow {}^7C_2 \times {}^6C_3$

Q:– How many ways 5 members can be selected from 6M & 7W

M^6	W^7	
5	0	$\longrightarrow {}^6C_5 \times {}^7C_0$
4	1	$\longrightarrow + {}^6C_4 \times {}^7C_1$
3	2	$\longrightarrow + {}^6C_3 \times {}^7C_2$
2	3	$\longrightarrow + {}^6C_2 \times {}^7C_3$
1	4	$\longrightarrow + {}^6C_1 \times {}^7C_4$
0	5	$\longrightarrow + {}^6C_0 \times {}^7C_5$

Sum $= {}^{13}C_5$

the committee

$\longrightarrow$ $^6C_3 \times {}^7C_2 + {}^6C_2 \times {}^7C_3 + {}^6C_1 \times {}^7C_4 + {}^6C_0 * {}^7C_5$

or Total → atmost 1 women. $= {}^{13}C_5 - ({}^6C_5 \times {}^7C_0 + {}^6C_4 \times {}^7C_1)$

Q:- 5 members committee formed from 7W & 6M.

(1) $P[\text{Committee has atleast } 2W] = \dfrac{{}^7C_2 \times {}^6C_3 + {}^7C_3 * {}^6C_2 + {}^7C_4 \times {}^6C_1 + {}^7C_5 \times {}^6C_0}{{}^{13}C_5}$

(or)

$P[\text{atleast } 2W] = 1 - P[\text{atmost } 1W]$

$= 1 - \dfrac{[{}^6C_5 * {}^7C_0 + {}^6C_4 \times {}^7C_1]}{{}^{13}C_5}$

Q:- 4 numbers are selected from 6 +ve & 8 -ve nos. What is the prob. that their product is +ve.

Soln:- Total possibilities → $^{14}C_4$

P (6)	N (8)	
4	0	$\longrightarrow$ ✓ $^6C_4 \times {}^8C_0$
3	1	
2	2	$\longrightarrow$ ✓ $^6C_2 \times {}^8C_2$
1	3	
0	4	$\longrightarrow$ ✓ $^6C_0 \times {}^8C_4$

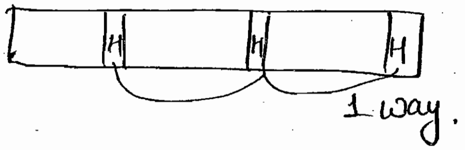
$P[\text{Product is +ve}] = \dfrac{{}^6C_4 \times {}^8C_0 + {}^6C_2 \times {}^8C_2 + {}^6C_0 \times {}^8C_4}{{}^{14}C_4}$

$= \dfrac{505}{1001}$

Q:- A coin is tossed n times. P[No. of heads = 3] = ?

P[getting 3 heads] = ?

Soln:- Total no. $= 2^n$, fav. 3 heads $= {}^nC_3 * 1$

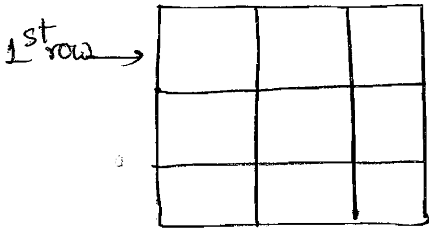


$P[\text{No. of heads}] = \dfrac{{}^nC_3}{2^n}$

P[the two squares have a side in common]

Sol^n :- Total no. of squares = 64

Two squares can be selected = $^{64}C_2$



3×3	8×8	
2	7	← 1st row
	7	← 2nd row
	⋮	
	7	← 8th row
	8*7	

3×3 2

8×8 $\underset{\substack{\uparrow \\ \text{1st colum}}}{7}$ $\underset{\substack{\downarrow \\ \text{2nd Col.}}}{7}$ 7 ---- $\underset{\substack{\uparrow \\ \text{8th Column}}}{7}$ = 8*7

No. of pair of squares having side in common = 7*8 + 8*7
= 56 + 56
= 112 Ans.

$\boxed{\text{22 Dec. 2014} \\ \text{L-7}}$

Q:- A coin is tossed n times,

P[Diff. b/w No. of Heads & no. of tails = n-2]

H	T	Diff.
n-1	1	n-2

= P[getting n-1 heads]

$$= \frac{^{n}C_{n-1} * 1}{2^{n}}$$

$^{n}C_{n-1}$ = Selecting n-1 places for n-1 heads out n places.

1 → Arranging n-1 heads in n-1 places.

Q:- A & B toss 3 coins simultaneously.

P[They get equal no. of heads].

→ If A & B toss 3 coins, Total outcomes = $\overset{A}{\underset{\substack{\downarrow \\ 2^3}}{\boxed{}}}$ $\overset{B}{\underset{\substack{\downarrow \\ 2^3}}{\boxed{}}}$

A & B get "0" heads = $\underset{\substack{\downarrow \\ ^3C_0}}{\boxed{}}$ $\underset{\substack{\downarrow \\ ^3C_0}}{\boxed{}}$

A & B get "1" heads = $\underset{\downarrow}{\boxed{}}$ $\underset{\downarrow}{\boxed{}}$

$${}^3C_2 * {}^3C_2 + {}^3C_3 * {}^3C_3$$
$$= 20 = 1+9+9+1$$

$$\therefore P[A \,\&\, B \text{ get equal heads}] = \frac{20}{2^3 * 2^3} = \frac{5}{16}$$

Permutations with constained repetitions —

$$P(n; q_1, q_2 ---- q_t) = n!/q_1! q_2! ---- q_t!$$

[Arrangement of n objects in which q_1 are alike, q_2 are alike, ------ q_t are alike.

$$q_1 + q_2 + ---- q_t = n]$$

Ex :- No. of arrangements of letters in the word MISSISSIPPI.

Solⁿ :— $P(11, \underset{S}{4}, \underset{I}{4}, \underset{P}{2}, \underset{M}{1}) = \frac{11!}{4!4!2!1!}$

a) so that all I's are together, $\boxed{4I's}$

1unit + 7unit = 8 unit

$$8 \text{ unit's} \longrightarrow P(8; \underset{S}{4}, \underset{P}{2}, \underset{M}{1}, \underset{4I's}{1}) = \frac{8!}{4!2!1!1!}$$

4I's can be arranged → 1

$$4I's = \frac{8!}{4!2!1!1!}$$

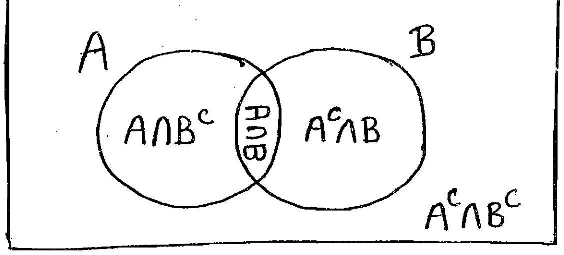
b) The letter of word MISSISSIPPI arranged.

$$P[\text{all } 4I\text{'s occur together}] = \frac{\left[\frac{8!}{4!2!1!1!}\right]}{\left[\frac{11!}{4!4!2!}\right]}$$

Addition Theorem :—

1) If A & B are any two events,

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$



2) $P(A^c \cap B^c) = 1 - P(A \cup B)$

Ex :- A box contain tickets numbered 1 to 20. A ticket is drawn at random.

1) P[Multiple of 5 or multiple of 7]
2) P[Multiple of 3 or multiple of 5]

Sol^n :- A = Multiple of 5; , B = Multiple of 7

$|A \cap B| = \left| \frac{20}{LCM(5,7)} \right| = 0 \rightarrow$ Disjoint event

$P(A \cup B) = P(A) + P(B) = \frac{4}{20} + \frac{2}{20} = \frac{6}{20} = \frac{3}{10}$

2) A = mult. of 3 , B = Mult. of 5

$|A \cap B| = \left[\frac{20}{LCM(5,3)} \right] = \left| \frac{20}{15} \right| = 1 \rightarrow$ Not disjoint
↳ have some value in set.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{\frac{20}{3} + \frac{20}{5} - \frac{20}{15}}{20} = \frac{9}{20}$

Cards :- (52 Cards)

♠ ♣ ♦ ♥ → 4 suits

A — — —
2
|
|
10
J — — —
Q
K

(i) $P[\text{King}] = {}^{4}C_1 / {}^{52}C_1 = 4/52 = 1/13$

(ii) $P[\text{ACE}] = {}^{4}C_1 / {}^{52}C_1 = 4/52 = 1/13$

(iii) $P[\text{King or Ace}] = P[K] + P[A] = \frac{{}^{4}C_1 + {}^{4}C_1}{{}^{52}C_1} = \frac{8}{52} = \frac{2}{13}$

(iv) $P[\text{King or Heart}] = P[K] + P[H] - P[H \cap K] = \frac{{}^{4}C_1 + {}^{13}C_1 - {}^{1}C_1}{{}^{52}C_1} = \frac{4}{13}$

Q:- Two cards are drawn from pack of 52 cards.

$P[\text{both are queen or both red}] = P[2Q] + P[2R] - P[2Q \cap 2R]$

$$P[2Q \cup 2R] = \frac{{}^{4}C_2 + {}^{26}C_2 - {}^{2}C_2}{{}^{52}C_2} = \frac{55}{221}$$

Q:- A box contains 125 bolts & 200 nuts. $\frac{1}{5}$th bolts & $\frac{3}{4}$th of nuts are defective. An item is selected from box. P[Def. or nut]

Sol^n :-

	B	N	Total
	125	200	= 325
Def:	$\frac{1}{5} \times 125$	$\frac{3}{4} \times 200$	
	25	150	= 175

150 → $D \cap N$

$P[\text{Def. or Nut}] = P[D] + P[N] - P[D \cap N]$

$$= \frac{175 + 200 - 150}{325}$$

$$= \frac{225}{325}$$

Conditional Probability -

Ex:-

3 red (R)
4 white (W)

1st draw : A ball is drawn at random.

Replaced

$P[R_2|R_1] = 3/7$

3R
4W

$P[R_2|NR_1] = 3/7$

(R_1 & R_2 are ~~not~~ independent)

Not Replace

4W
2R

→ $P[R_2|R_1] = 2/6 = 1/3$

3R
3W

$P[R_2|NR_2] = 3/6 = 1/2$

Dependent events

does not effect happening or non-happening of other event then th events are said to be independent, otherwise they are dependent

Conditional Prob. :—

- $P[A|B] \rightarrow$ Prob. of A given B has already occured.

$$P[A|B] = \frac{P(A \cap B)}{P[B]}, \quad P(B) \neq 0$$

$$P[B|A] = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0$$

Ex :— A die is rolled once,

P[multiple of 3 given that a no. ~~of~~ greater than 4 has occured]

$$= \frac{1}{2}$$

Method1:— we know directly as $>4 = 5 \,\&\, 6$ out of which only 6 i multiple of 3 i.e. getting 6. It has prob. 1/2 (~~one~~

Method2 :— A = Multiple of 3 = {3, 6} , B = ~~Multi~~ Greater than 4={

$A \cap B$ = Multiple of 3 & greater than 4 = {6}

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{2/6} = \frac{1}{2}$$

Multiplication Theorem :—

$$P(A \cap B) = P(A) \cdot P(B/A)$$
$$= P(B) \cdot P(A|B)$$

Results —

1) If A & B are independent events,

(i) $P(A/B) = P(A)$

(ii) $P(B/A) = P(B)$

(iii) $P(A \cap B) = P(A) \cdot P(B)$

(iv) A^c, B are also independent ; $P(A^c \cap B) = P(A^c) \cdot P(B)$

6) A^c, B^c are –"– –"– ——

Q:– The prob. of solving a prblm by 3 students X, Y & Z are
$P[X] = 1/4$, $P[Y] = 3/7$, $P(Z) = 2/9$.
If all of them try independently then what is the prob. that prblm cannot be solved. ~~or~~ & (solved or atleast one solve prblm)

Sol^n:– $P[X^c] = 3/4$, $P[Y^c] = 4/7$, $P(Z^c) = 7/9$

$$P[\text{cannot solved}] = P[X^c \wedge Y^c \wedge Z^c] = P[X^cY^cZ^c]$$
$$= P[X^c]\,P[Y^c]\,P[Z^c] = \frac{3}{4} \times \frac{4}{7} \times \frac{7}{9} = \frac{1}{3}$$

$$P[\text{can be solved}] = P[\text{at least one solve the prblm}]$$
$$= P[X \cup Y \cup Z] = 1 - P[\text{none}]$$
$$= 1 - 1/3 = 2/3$$

⟶

(Q) Let $P(G_1) = 0.5$, $P(G_2) = 0.6$, $P(G_3) = 0.8$ where G_1, G_2, G_3 are targets hit by gun G_1, G_2 G_3 resp. Let G_1, G_2 & G_3 are independent then what is the prob that atleast two hits registered.

Sol^n:– $P[\text{atleast two hits registered}] = P[G_1G_2G_3^c] + P[G_1G_2^cG_3] + P[G_1^cG_2G_3] + P[G_1G_2G_3]$

$$= 0.5 \times 0.6 \times 0.2 + 0.5 \times 0.4 \times 0.8 + 0.5 \times 0.6 \times 0.8 + 0.5 \times 0.6 \times 0.8$$
$$= 0.7$$

Q:– A ~~pair of~~ fair dice rolled twice. What is the prob. that an odd no. will follow an even no.

Sol^n:– $\underset{P_e}{\square}\,\underset{P_o}{\square}$ $\quad = \underset{P_e \cdot P_o}{\frac{3}{6} \cdot \frac{3}{6}} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

Q:– A fair coin is tossed 10 times. What is the prob. that only 1st two tosses will yield heads.

$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2^{10}}$ Ans.

Q:— A coin is tossed again & again until head appears for 1st time. P[odd no. of tosses reqd.] or P[No. of tosses reqd. is odd]

Soln:— ① or ①②③ or ①②③④⑤ ----

$\underset{H}{①}$ or $\underset{T\ T\ H}{①②③}$ or $\underset{T\ T\ T\ T\ H}{①②③④⑤}$

$$= \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\frac{1}{2}\cdot\frac{1}{2}\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}\left[1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+ ---\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{1}{4}}\right] = \frac{2}{3} \text{ Ans.}$$

$$P[\text{no. of tosses reqd. is even}] = 1 - P[\text{no. of tosses} = \text{odd}]$$
$$= 1 - \frac{2}{3} = \frac{1}{3}$$

Q:— A pair of fair dice rolled again & again till a total of 5 or 7 is obtained. What is the prob. that a total of 5 comes before total of 7.

Soln:—

P[total 5 comes before total 7]

$$P[S=5] = \frac{4}{36}, \quad P[S=7] = \frac{6}{36}$$

$$P[S=5 \text{ or } S=7] = \frac{10}{36}$$

$$P[S\neq 5 \ \& \ S\neq 7] = 1 - P[S=5 \text{ or } S=7]$$
$$= 1 - \frac{10}{36} = \frac{26}{36}$$

To get $S=5$ before $S=7$

$\underset{S=5}{①}$ or $\underset{S\neq5\ \&\ S\neq7}{①}\ \underset{S=5}{②}$ or $\underset{S\neq5\ \&\ S\neq7}{①}\ \underset{S\neq5\ \ S\neq7}{②}\ \underset{S=5}{③}$ or ---

$$\frac{4}{36} + \frac{26}{36}\cdot\frac{4}{36} + \frac{26}{36}\cdot\frac{26}{36}\cdot\frac{4}{36} + ---$$

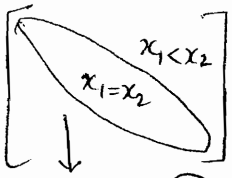
$$\frac{4}{36}\left[1+\frac{26}{36}+\left(\frac{26}{36}\right)^2 + ---\right] = \frac{4}{36}\frac{1}{1-\frac{26}{36}} = \frac{\cancel{36}\times 4}{\cancel{36}\times 10} = 0.4 \text{ or } \frac{2}{5}$$

of getting sum = 7 when it is known that digit in the 1st die is greater than that of 2nd one.

Sol^n:— Digit on 1st die x_1,
Digit on 2nd die x_2.

$$P\left[S=7 \,/\, (x_1 > x_2)\right] = \frac{3}{15} = \frac{1}{5}$$

$S=7 \rightarrow$ ~~(1,6)~~, ~~(2,5)~~, ~~(3,4)~~ ; (4,3), (5,2), (6,1)



$x_1 > x_2 = \boxed{15}$

Q:— Bag P contain 3W & 4B balls
Bag Q contain 4W & 3B balls.
A ball is transferred at random from P to Q & then a ball is transferred at random from Q to P. A ball is then taken from P. What is the chance that it is a white ball.

Sol^n:— 3W 4B (P) 4W 3B (Q)

① ② ③ or ① ② ③ or ① ② ③
W W W   W B W   B W W

$= 3/7\ 5/8\ 3/7$ 1st case $+ 3/7\ 3/8\ 2/7 + 4/7\ 4/8\ 4/7$

or ① ② ③
B B W

$+ \frac{4}{7}\frac{4}{8}\frac{3}{7}$

3W ~~2W~~ ~~3W~~ 4B   4W ~~5W~~ ~~4W~~ 3B

1st case.

$$= \frac{45+18+64+48}{7\cdot8\cdot7} = \frac{\cancel{175}^{25}}{\cancel{7}\cdot7\cdot8}$$

$= \frac{25}{56}$ Ans.

P-38: Q-48

2I correct   4D wrong

① ② or ① ②
Correct key is lost → lock is opened (Correct key selected)
wrong key lost   lock is opened (Correct key is selected)

$$\frac{2}{6}\cdot\frac{1}{5} + \frac{4}{6}\cdot\frac{2}{5} = \frac{2+8}{30} = \frac{1}{3}$$

Q-30 : P-36 : $P(A) = X$, $P(B) = Y$

$P(A^C) = 1-X$, $P(B^C) = 1-Y$

A & B both agree = Both true or Both false

$= AB$ or A^CB^C
$= XY + (1-X)(1-Y)$

$$P(\text{statement true} \mid \text{Both agree}) = \frac{XY}{XY+(1-X)(1-Y)}$$

$XY \rightarrow$ Statement true & both agree.

Q:- Parcels from sender S to receiver R pass sequentially through two post offices. Each post office has prob. ~~of~~ 1/5 of losing an incoming parcel, independly of all other parcels. Given a parcel is lost, prob. it was lost by second post office is

Sol^n:— $P[\text{Lost by 1st}] = 1/5$

$P[\text{Lost by } II^{nd}] = \frac{4}{5}\cdot\frac{1}{5} = \frac{4}{25}$

i.e $P[$~~lost~~ not lost by 1st$]$ & $P(\text{lost by } II^{nd})$

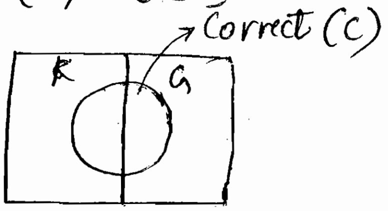
$\frac{4}{25}$ ~~Ans~~.

$P[\text{lost}] = P[\text{Lost by 1st or Lost by } II^{nd}]$
$= \frac{1}{5} + \frac{4}{5}\cdot\frac{1}{5}$

$$P[\text{Lost by II} / \text{Lost}] = \frac{4/5 \cdot 1/5}{\frac{1}{5}+\frac{4}{5}\frac{1}{5}} = \frac{4}{9}$$

knows (K) | Guess (G)

$P(K) = 0.7$ } Priori
$P(G) = 0.3$ } Probabilities



Likelihoods

Sol^n :– $P(C/K) = 1$ $\quad P(C/G) = \frac{1}{4} \rightarrow$ (4 options; one is correct)

$P[\text{Correct}] = P[K \cap C \text{ or } G \cap C]$

Total Prob.

$= P[K \cap C] + P[G \cap C] = P(K)\,P(C/K) + P(G)\,P(C/G)$

$= 0.7 \times 1 + 0.3 \times \frac{1}{4} = 0.7 + 0.3 \times 0.25 = 0.775.$

Now, $P[K/C]$, $P[G/C]$ } Posterior Probabilities or Bayes Probabilities

$$P(K/C) = \frac{P[K \cap C]}{P[C]} = \frac{0.7 \times 1}{0.775} = 0.9039$$

$$P[G/C] = \frac{P[G \cap C]}{P[C]} = \frac{0.3 \times \frac{1}{4}}{P[C]} = 0.0967$$

Ex :– There are three companies X, Y & Z supply computers to a university

Company	% computers supplied	Prob. of being defective.
X	60 %	0.01
Y	30 %	0.02
Z	10 %	0.03

1) $P[\text{Comp being Defective}] \rightarrow$ Total Prob.

2) $P[X|D]$, $P[Y|D]$, $P[Z|Z]$ } Posterior or Bayes Prob.

$P(X) = 0.6$, $P(Y) = 0.3$ $P(Z) = 0.$

$P(D/X) = 0.01$, $P(D/Y) = 0.02$

$P(D/Z) = 0.03$

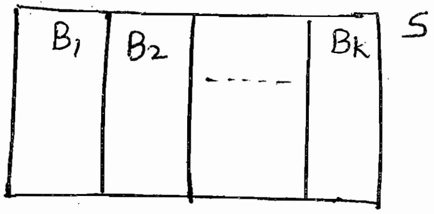
1) $P[\text{Defected}] = 0.6 \times 0.01 + 0.3 \times 0.02 + 0.1 \times 0.03 = 0.006 + 0.006 + 0.003$
$= 0.015$

2) $P[X|D] = \frac{0.6 \times 0.01}{0.015} = 0.4$

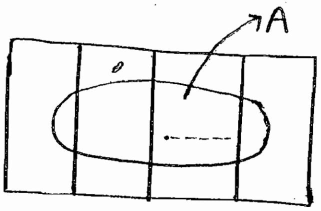
$P[Z|D] = \frac{0.1 \times 0.03}{0.015} = \cancel{0}\,\frac{.\cancel{3}1}{\cancel{15}5} = 0.2$

$P[Y|D] = \frac{0.3 \wedge 0.02}{0.015} = 0.4$

Let $B_1, B_2, \text{------} B_K$ the events which partition S.



Let A be any event following these events



The total prob. of A;

$P(A) = P(B_1) \cdot P(A/B_1) + P(B_2) P(A|B_2) + \text{-------} P(B_K) \cdot P(A|B_K)$

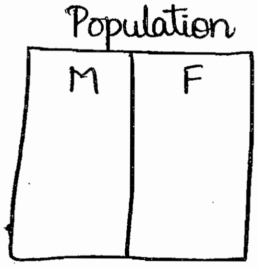
(or)

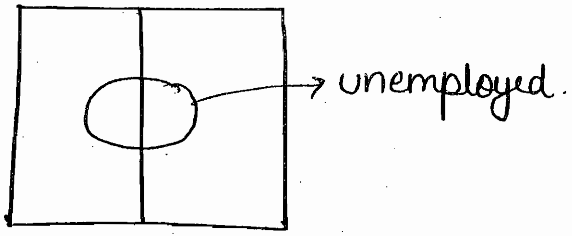
$$\boxed{P(A) = \sum_{i=1}^{K} P(B_i) P(A|B_i)}$$

<u>Bayes Theorem</u> :- Let $P(A) \neq 0$

$$P(B_j/A) = \frac{P(A \cap B_j)}{P(A)} \Rightarrow \boxed{P(B_j/A) = \frac{P(B_j) P(A|B_j)}{P(A)}}$$

<u>Ex</u> :-





$P(M) = 0.7$ $\qquad$ $P(U/M) = 0.25$

$P(F) = 0.3$ $\qquad$ $P(U|F) = 0.15$

$P(U) = 0.7 \times 0.25 + 0.3 \times 0.15$

$P(M/U) = \frac{0.7 \times 0.25}{P(U)}$; $P(F/U) = \frac{0.3 \times 0.15}{P(U)}$

Random Variable (RV) — (Main purpose → outcome as nos)

$X : S \rightarrow R$

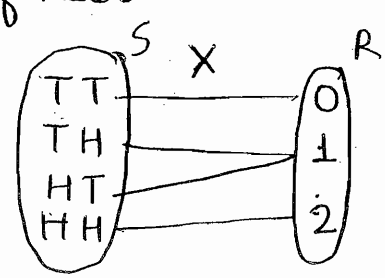
It associates real no. to outcomes of the sample space.

R.V

↓ Discrete RV
→ take discrete values

↓ Continuous RV
→ take values in the interval.

X : No. of heads.



$X = \{0, 1, 2\}$

$X = x_1$	x_2	------	x_n
$P(x) = P(x_1)$	$P(x_2)$	------	$P(x_n)$

Discrete Prob. distribution

Ex :- Working of a bulb ; X : No. of hrs bulb works before failure

$X : [0, \infty)$

$P(a \le X \le b)$; we can find continuous prob. distributions generally expressed graphs.

Prob. Mass Functions (PMF) — If X is discrete R.V. then its pmf is defined as $P(X=x) = p(x)$ & pmf is such that (1) $p(x) \ge 0$ (2) $\sum_x p(x) = 1$

Ex :- Tossing a coin twice — X : no. of heads

X :	0	1	2
$P(x)$	1/4	2/4	1/4

$P(X=0) = p(0) = 1/4$

$P(X=1) = p(1) = 2/4$

$P(X=2) = p(2) = 1/4$

Ex :- A R.V. has the following distribution

x	0	1	2	3	4	5	6	7
p	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

$\Sigma P = 1$

$= 10k^2 + 9k = 1$

(i) $k = 1$ (ii) $P(X > 2)$

~~k(10k~~ $10k^2 + 9k - 1 = 0$

$\frac{-9 \pm \sqrt{81+40}}{20} = \frac{-9 \pm 11}{20}$

$= 0.1$ ~~or~~

(-ve is neglected

$P(X > 2) = 1 - P(X \le 2)$

$= 1 - [0 + k + 2k] = 1 - 3k = 1 - \frac{3}{10} = 0.7$

Q :- A die is loaded so that Prob of getting face x is α to x. P[getting odd no. when the die rolled]

Sol^n :-

$P[x] \propto x$

$P[x] = kx$

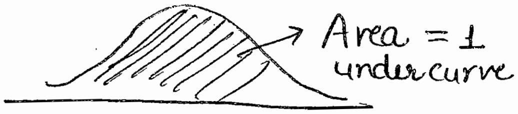
X	1	2	3	4	5	6
$P(X)$	k	2k	3k	4k	5k	6k

fair (unbiased) coin	Biased Coin
$P(H) = 1/2$ $P(T) = 1/2$	$P(H) = 3/4$ $P(T) = 1/4$
dice : $P(1) = \_\_ = P(6) = 1/6$	

$P[\text{odd no.}] = P\{1 \text{ or } 3 \text{ or } 5\} = P(1) + P(3) + P(5) = k + 3k + 5k$
$= \frac{9}{21} = \frac{3}{7}$

Prob. density f^n (pdf) – If X is a cont. R.V. then its prob. density f^n $f(x)$ is defined such that,

(i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)dx = 1$



(iii) $P(a \leq X \leq b) = \int_a^b f(x)dx$

Q:- A cont. R.V. X has pdf $f(x)$ given by

$$f(x) = \begin{cases} \lambda e^{-x/100} & x \geq 0 \\ 0 & o|w \end{cases}$$

i) $\lambda = ?$
(ii) $P(50 < X < 100)$

<u>Sol^n</u> :— (i) $\int_0^{\infty} \lambda e^{-x/100} dx = 1 \Rightarrow \frac{-\lambda}{1/100}[e^{-x}]_0^{\infty} = 1 \Rightarrow \lambda = 1/100$

(ii) $\int_{50}^{100} \frac{1\cancel{00}}{100} e^{-x/100} dx = \frac{1}{100}\left[\frac{e^{-x/100}}{-1/100}\right]_{50}^{100}$
$= -1\left[e^{-1} - e^{-1/2}\right] = e^{-1/2} - e^{-1}$ <u>Ans</u>.

<u>Statistics</u> :— Characterizes the distribution.

Measures of:

1) <u>Central</u> <u>tendency</u> :– Central value
Mean, Median, Mode

2) <u>Dispersion</u> : Spread of the data.
Variance $= \sigma^2$, Std Deviation $= \sigma$

<u>For ungrouped data</u> : $x_1, x_2, ----x_n$

<u>Mean</u> $= \mu = \frac{x_1 + x_2 + ------ x_n}{n}$

<u>Median</u> → Sort the data
Median = mid value or avg. of mid value.

Ill defined Distribution : $\boxed{\text{Mode} = 3\,\text{Median} = 2\,\text{Mean}}$

Variance, $\sigma^2 = \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{n}$

Avg of squared deviations from mean :

Std deviation – S.D. = Positive square root of variance $= \sigma$

Q:- 6, 2, 1, 2, 3, 4. Find mean, median, mode, σ^2, σ

Solⁿ:- Mean $= \frac{6+2+1+2+3+4}{6} = 3$

For median : 1, 2, 2, 3, 4, 6 $\longrightarrow \frac{2+3}{2} = 2.5$

Mode : 2

Variance : $\sigma^2 = \frac{(6-3)^2+(2-3)^2+(1-3)^2+(2-3)^2+(3-3)^2+(4-3)^2}{6} = \frac{8}{3}$

$\sigma = \sqrt{\frac{8}{3}}$

Mathematical Expectation:- Or expected value → gives averages.

$$E[X] = \begin{cases} \sum_x x\,p(x) & X \text{ is discrete R.V.; } p(x) \text{ is Pmf} \\ \int_{-\infty}^{\infty} x f(x)\,dx & X \text{ is } \underset{\hookrightarrow \text{Cont.}}{\text{C.R.V.}} ;\ f(x) \text{ is pdf.} \end{cases}$$

1) Mean $= \mu = E[X]$

* $$E[X^2] = \begin{cases} \sum_x x^2 p(x) & X \text{ is D.R.V.} \\ \int_{-\infty}^{\infty} x^2 f(x)\,dx & X \text{ is C.R.V.} \end{cases}$$

Variance : $\text{Var}(X) = E[(X-\mu)^2]$

Result –

1. $E[C] = C$ 2. $E[aX+b] = aE[X]+b$ 3. $E[X+Y] = E[X]+E[Y]$

9. $E[X \cdot Y] = E[X] \cdot E[Y]$ iff X & Y are independent R.V.

***5. $\text{Var}(X) = E[X^2] - \mu^2$ 6. $\text{Var}(C) = 0$

7. $\text{Var}(aX+b) = a^2\,\text{Var}(X)$ 8. $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$ iff X & Y are independent R.V.

ance of the following distribution.

x	0	1	2
$P(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Sol^n:- $\mu = E[X] = \sum_{x} x p(x) = 0 * \frac{1}{4} + 1 * \frac{2}{4} + 2 * \frac{1}{4}$

$\Rightarrow \mu = 1$

Variance $= E[x^2] - 1$

x^2	0	1	4	→ Square only x
$P(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	$P(x)$ remains same.

$E[x^2] = \sum x^2 p(x)$

$= 0 * \frac{1}{4} + 1 * \frac{2}{4} + 4 * \frac{1}{4} = \frac{3}{2}$

$\text{Var.}(X) = \frac{3}{2} - 1^2 = \frac{1}{2} = \sigma^2$; S.D., $\sigma = 1/\sqrt{2}$

Continuous Distribution -

Uniform distribution - A cont. R.V. x is said to follow uniform distribution over the interval $[a,b]$ if its pdf $f(x)$ is given by $\boxed{f(x) = \frac{1}{b-a} \quad a \leq x \leq b}$

$\text{Mean} = \mu = E[X] = \int_{-\infty}^{\infty} x f(x) dx$

$\boxed{\text{Mean} = \mu = \frac{a+b}{2}}$ $= \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \frac{x^2}{2} \Big|_a^b = \frac{a+b}{2}$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a} \frac{x^3}{3} \Big|_a^b = \frac{b^3 - a^3}{3(b-a)}$

$= \frac{b^2 + ab + a^2}{3}$

$\text{Variance} = \frac{a^2 + ab + a^2}{3} - \left(\frac{a^2 + 2ab + b^2}{4}\right) = \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$

$\boxed{\text{Variance}(x) = \frac{(b-a)^2}{12}}$, $\boxed{\text{S.D.} = \frac{b-a}{\sqrt{12}}}$

uniformly at random. Then the expected length of shorter stick is ______

Sol^n:- Let X : Length of shorter stick.

X is uniformly distributed b/w 0 to 1/2

pdf, $f(x) = \frac{1}{1/2-0} = [0, 1/2]$

$$f(x) = 2 \qquad 0 \le x \le 1/2$$

$$\mu = E[X] = \frac{a+b}{2} = \frac{1}{4} = \left(\frac{0+1/2}{1/2}\right)$$

Q:- The life of electric bulb is a R.V. with exp. distribution

$$f(t) = \alpha e^{-\alpha t} \qquad t \ge 0$$

The prob. that its value lies b/w 100 hrs to 200 hrs.

a) $e^{-100\alpha} - e^{-200\alpha}$ b) $e^{-200\alpha} - e^{-100\alpha}$

c) $e^{-100} - e^{-200}$ d) $e^{-200} - e^{-100}$

Sol^n:- $P(100 < T < 200) = \int_{100}^{200} \alpha e^{-\alpha t}\, dt = \cancel{\alpha}\frac{e^{-\alpha t}}{-\cancel{\alpha}}\Big|_{100}^{200} = e^{-100\alpha} - e^{-200\alpha}$

Exponential Distribution :-

A cont. R.V. $X \sim E(\lambda)$ [X follows exp. distribution with parameter λ]

If pdf is given by $f(x) = \lambda e^{-\lambda x}$ $\quad x \ge 0$

Mean = $\mu = \frac{1}{\lambda}$, Variance $= \sigma^2 = \frac{1}{\lambda^2}$

Normal Distribution:- (Gaussian distribution)

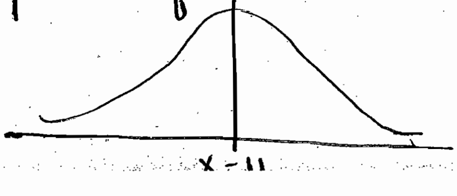
A. cont. R.V. $X \sim N(\underset{\substack{\downarrow \\ \text{Mean}}}{\mu}, \sigma^2) \rightarrow \text{Variance}$

if pdf is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty \text{ to } \infty$$

Properties of Normal Distribution-

1)



1) Normal curve is bell shaped.

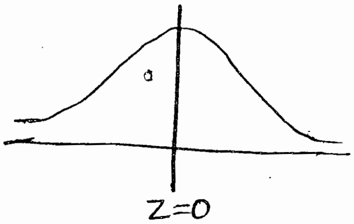
2) -"- -"- is symmetrical abt. mean $(X = \mu)$

3) $P(X>\mu) = 0.5$ 4) $P(X<\mu) = 0.5$

∴ Mean = Median = Mode

Standard normal variate — $Z = \frac{X-\mu}{\sigma}$

Def :- The std normal variate $Z \sim N(\text{Mean}=0, \text{variance}=1)$ & its pdf is given by $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ $-\infty < Z < \infty$



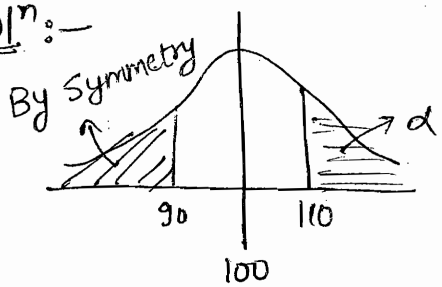
1) Standard Normal Curve is symmetric abt $Z=0$.

2) $P(Z<0) = P(Z>0) = 0.5$

3) The area under the curve from 0 to Z_1 [i.e $P(0<Z<Z_1)$] is tabulated in std normal tables.

Ex :- X is a normal R.V. with mean 100. $P(X \geq 110) = \alpha$, then $P(90 < X < 110)$ a) $1-\alpha$ ~~b) $1-2\alpha$~~ c) 2α d) $1-\alpha/2$

Soln :-



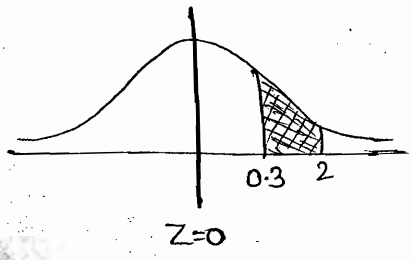
$P(90<X<110) = 1-2\alpha$

$= 1 - P(X<90) - P(X>110)$

Q :- Suppose that temperature during december is normally distributed with mean = 20° & std deviation, $\sigma = 3.33°$. Find ~~prob (21.11°~~ $P(21.11° < X < 26.66°)$. Area under the curve $z=0$ to $z=2$ is 0.4772. $Z=0$ to $Z=0.33$ is 0.1293

Soln :- $Z = \frac{X-\mu}{\sigma}$

$$P\left(\frac{21.11-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{26.66-\mu}{\sigma}\right) = P(0.3 \leq Z \leq 2)$$



$P(0<Z<2) - P(0<Z<0.3)$

$= 0.4772 - 0.1293$

Bernoulli's Trials :— Independent trials of repetitive nature in which prob. (success or failure) of a particular event does not change from one trial to next trial.

Binomial Distribution :— A discrete R.V. X is said to follow binomial distribution with parameters n, p if its pmf is given by

$$\boxed{P(X=r) = {}^{n}C_{r}\, p^{r} q^{n-r}} \qquad r = 0, 1, 2 \text{-----} n$$

$n \to$ No. of Bernoulli's trial.

$p \to P(A) =$ Prob. of success of an event.

$q \to P(A^{c})$

R.V $\to$ X : No. of times A happens.

$P(X=r) \to$ Prob. of event A happens r times (or) prob. of r successes of event A.

Mean = np

Variance = $\sigma^2 = npq$

A. R.V. which follows binomial distribution is called binomial R.V.

Q:- A coin is tossed 8 times (1) What is the prob. of getting 3 heads.
(2) Prob. of getting atleast 2 heads.,

Solⁿ:— A : getting heads ; $P = P(A) = 1/2$; $q = 1/2$

X : No. of times A happens.

(i) $P(X=3) = {}^{8}C_{3}\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{5} = {}^{8}C_{3}\left(\frac{1}{2}\right)^{8} = \frac{{}^{8}C_{3}}{2^{8}}$

(ii) $P(X \geq 2) = 1 - P[X \leq 1] = 1 - [P(0) + P[1]] = 1 - \left(\frac{{}^{8}C_{0} + {}^{8}C_{1}}{2^{8}}\right)$

$= 1 - \frac{(1+8)}{2^{8}} = 1 - \frac{9}{2^{8}} = \frac{247}{256}$

Q:- A coin is loaded in such a way that prob. of getting heads is 3 times prob. of getting tails. ~~If~~ If the coin is tossed 4 times, what is the prob. of getting atleast 3 heads.

Solⁿ:— $P(H) = 3P(T)$, Let $P(T) = x$

$P(H) = 3x$

$x + 3x = 1 \Rightarrow x = 1/4$

$p = 3/4$ (getting head) , $q = 1/4$

$P(X \geq 3) = P(3) + P(4) = {}^{4}C_{3}\left(\frac{3}{4}\right)^{3}\left(\frac{1}{4}\right) + {}^{4}C_{4}\left(\frac{3}{4}\right)^{4} = \frac{4 \cdot 3^{3}}{4 \cdot 4^{3}} + \frac{3^{4}}{4^{4}}$

44 256

Q:- A die has four blank faces & 2 faces marked 3. What is chance of getting 12 in 5 throws.

Solⁿ:- $P[0] = 4/6 = 2/3$; $P[3] = \frac{2}{6} = \frac{1}{3}$

P[getting 12 in 5 throws] = P[4 3's in 5 throws]

$\underline{n=5}$; p = prob. getting 3 = 1/3 ; $q = 2/3$

$$P[X=4] = {}^5C_4\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)^1 = 5 \cdot \frac{2}{3^5} = \frac{10}{243}$$

Q:- A man takes step frwd with prob. 0.4 & step bckwrd with prob 0.6. The prob. that at the end of 11 steps, he is one step away from starting pt. is ____

Solⁿ:- $n = 11$; p = prob. of taking frwd steps = 0.4

q = prob ——〃—— bckward step = 0.6

$$P[\text{one step away}] = P[\text{one step frwd or one step bckwrd}]$$
$$= P[\text{one step F}] + P[\text{one step B}]$$
$$= P[6F, 5B] + P[5F, 6B]$$
$$= P[X=6] + P[X=5]$$
$$= {}^{11}C_6(0.4)^6(0.6)^5 + {}^{11}C_5(0.4)^5(0.6)^6$$

$\because$ ${}^nC_r = {}^nC_{n-r}$; ${}^{11}C_6 = {}^{11}C_5$

$$= {}^{11}C_6(0.4)^5(0.6)^5\left[\underbrace{0.4+0.6}_{1}\right]$$
$$= {}^{11}C_6(0.4)^5(0.6)^5$$

Q:- An Unbiased coin tossed ∞ times. What is the prob. that the 4th head appears at 10th toss.

a) 0.067 b) 0.073 [✓ c) 0.082] d) 0.091

Solⁿ:-

$\boxed{9 \text{ tosses } 3H}$ ⑩
H

$P(X=3)\cdot\left(\frac{1}{2}\right) = {}^{9}C_3\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)\cdot\frac{1}{2} = \frac{{}^{9}C_3}{2^{10}} = \frac{21}{256} = 0.082$

↳ at 10th toss

Q:– The prop. of a man hitting a target is 1/4. If he fires 4 times th prob. of hitting target atleast twice.

Sol^n:– $n=4$; $p = 1/4$, $q = 3/4$

$$P(X \geq 2) = 1 - [P(X \leq 1)] = 1 - [P(0) + P(1)] = 1 - \left[{}^{4}C_0\left(\frac{1}{4}\right)^{0}\left(\frac{3}{4}\right)^{4} + {}^{4}C_1\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{3}\right]$$

$$= 1 - \left[\frac{3^4 + 4\cdot3^3}{4^4}\right] = \frac{6^7}{4^4} = 0.26$$

Binomial Freq. Distribution – An experiment consists of n bernoulli's trials & the experiment is repeated N times.

The frequency of r successes (Expected value of r successes)

$$= \boxed{N * {}^{n}C_r\, p^r q^{n-r}}$$

Q:– In 256 sets of 12 tosses of a coin. How many cases one can expect 8 heads.

Sol^n:— $N = 256$; $n=12$, $p = 1/2$, $q = 1/2$.

Expected ~~of~~ value of 8 heads in total 256 $= N * {}^{n}C_r\, p^r q^{n-r}$.

$$= 256 * P(X=8) = 256 * {}^{12}C_8\left(\frac{1}{2}\right)^{8}\left(\frac{1}{2}\right)^{4}$$

$$= \frac{256 * {}^{12}C_8}{2^{12}} = \frac{{}^{12}C_8}{2^4} = \frac{495}{16} = 30.9 \approx 31.$$

Poisson Distribution :– These are the distributions related to events which are extremely rare.

Limiting case of binomial distribution when no. of trials n is (∞) very large; & p is very small.

Def:– A discrete R.V $X \sim P(\lambda)$ is said to follow poisson distribution if its pmf is given by

$$\boxed{P(X=r) = \frac{e^{-\lambda}\lambda^r}{r!}} \quad r = 0,1,2------$$

Mean $= \lambda$

Variance $= \lambda$

$X \sim B(n,p)$ Put $\boxed{\lambda = np}$ when $n \to \infty$ & p is very small

$\approx X \sim P(\lambda)$

Q:– The prob. of a bad reaction from certain injection is 0.001. What is the chance that out of 2000 individuals more than two will get a bad reaction.

Sol^n:– $n = 2000$; $p = 0.001$; $n \to \infty$ & p is very small

$\lambda = np = 2$

$X \sim B(n,p) \approx \cancel{X} \sim P(2)$ $\qquad P(X=r) = \dfrac{e^{-\lambda}\lambda^r}{r!}$ $\quad r = 0, 1, 2 \ldots$

P[more than two will get bad reaction] $= P[X > 2] = 1 - P[X \leq 2]$

$= 1 - \{P[X=0] + P[1] + P[2]\}$

$= 1 - \left[\dfrac{e^{-2}2^0}{0!} + \dfrac{e^{-2}2^1}{1!} + \dfrac{e^{-2}2^2}{2!}\right] = 1 - \dfrac{5}{e^2}$ Ans.

Q:– A traffic office imposes on an avg 5 penalties daily on traffic violators. Assume that no. of penalties on different days is independent & follows poisson's distribution. What is the prob. that there will be less than 4 penalties on a day. $\because \lambda = 5$

Sol^n:– $P[X<4] = P(0) + P(1) + P(2) + P(3)$

$= \dfrac{e^{-5}5^0}{0!} + \dfrac{e^{-5}5^1}{1!} + \dfrac{e^{-5}5^2}{2!} + \dfrac{e^{-5}5^3}{3!} = e^{-5}\left[1 + 5 + \dfrac{25}{2} + \dfrac{125}{6}\right]$

$=$ 0.265 Ans.

DEFINITE INTEGRAL

1) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2) $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

3) $f(-x) = f(x)$; $f(x)$ is even f^n.

4) $f(-x) = -f(x)$; $f(x)$ is odd f^n.

5) $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ even $f(x)$

$= 0$ $\quad f(x)$ odd

6) $\displaystyle\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$; $\quad f(2a-x) = f(x)$

$= 0$; $f(2a-x) = -f(x)$

$$\int_0^{\pi}\cos x\,dx = 0 \quad \because \cos(\pi - x) = -\cos x$$

Q:-1. $\int_{-\pi/2}^{\pi/2} x^{10}\log\left(\frac{1+\sin x}{1-\sin x}\right)dx$ 2. $\int_0^{\pi/2}\left(a^2\cos^2 x + b^2\sin^2 x\right)dx$

3. $\int_0^{\pi/2}\frac{\cos x - \sin x}{1+\cos x\sin x}dx$ 4. $\int_0^{\pi}\frac{1}{a^2\cos^2 x + b^2\sin^2 x}dx$

5. $\int_0^{\pi/4}\log(1+\tan x)dx$ 6. $\int_0^{\pi/4}\frac{1-\tan x}{1+\tan x}dx$

Sol^n:—1. $f(-x) = (-x^{10})\log\left(\frac{1+\sin(-x)}{1-\sin(-x)}\right) = x^{10}\log\left(\frac{1-\sin x}{1+\sin x}\right)$

$= -x^{10}\log\left(\frac{1-\sin x}{1+\sin x}\right) \longrightarrow -f(x)$ odd f^n.

$\int f(x) = 0$ Ans.

2. $I = \int_0^{\pi/2}\left(a^2\cos^2 x + b^2\sin^2 x\right)dx$ —— ①

$I = \int_0^{\pi/2}\left[a^2\cos^2\left(\frac{\pi}{2}-x\right) + b^2\sin^2\left(\frac{\pi}{2}-x\right)\right]dx = \int_0^{\pi/2}\left(a^2\sin^2 x + b^2\cos^2 x\right)dx$ —— ②

① + ② $= 2I = \int_0^{\pi/2}(a^2+b^2)\,dx = (a^2+b^2)x = (a^2+b^2)\frac{\pi}{2}$

$I = \frac{\pi}{4}(a^2+b^2)$ Ans.

3. $I = \int_0^{\pi/2}\frac{\cos\left(\frac{\pi}{2}-x\right) - \sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)} = \int_0^{\pi/2}\frac{\sin x - \cos x}{1+\cos x\sin x}dx \longrightarrow$ ②

$2I = 0 \Rightarrow I = 0$

4. $I = \int_0^{\pi}\frac{1}{\cos^2 x\left[a^2+b^2\tan^2 x\right]}dx = \int_0^{\pi}\frac{\sec^2 x}{a^2+b^2\tan^2 x}dx$

Rule ⑥ $f(2a-x) = f(x)$ $\sec^2(\pi - x) = (-\sec x)^2 = \sec^2 x$

$\tan^2(\pi - x) = (-\tan x)^2 = \tan^2 x$

Put $\tan x = t$ $x = 0$ $t = 0$

$\sec^2 x\,dx = dt$ $x \to \frac{\pi}{2}$ $t \to \infty$

$$\int_0 \frac{}{a^2+b^2t^2} \quad = \frac{}{b^2}\int_0 \frac{}{(a/b)^2+t^2}$$

$$= \frac{2}{b^2}\times\left(\frac{b}{a}\right)^a \left|\tan^{-1}\frac{ta}{b}\right|_0^\infty = \frac{2}{ab}\left[\frac{\pi}{2}-0\right]$$

$$I = \frac{\pi}{ab} \quad \underline{\underline{\text{Ans}}}.$$

5. $I = \int_0^{\pi/4} \log(1+\tan x)\,dx$ Rule ②, $I = \int_0^{\pi/2} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$

$$I = \int_0^{\pi/4} \log\left(1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)dx = \int_0^{\pi/4} \log\left[\frac{1+\cancel{\tan x}+1-\cancel{\tan x}}{1+\tan x}\right]dx$$

$$2I = \int_0^{\pi/4} \log 2\,dx - \int_0^{\pi/4} \log(1+\tan x)\,dx$$

$$2I = \log 2\,[x]_0^{\pi/2} \Rightarrow I = \frac{\pi}{8}\log 2 \quad \underline{\underline{\text{Ans}}}.$$

~~6. $I = \int_0^{\pi/4} \frac{1-\tan\left(\frac{\pi}{4}-x\right)}{1+\tan\left(\frac{\pi}{4}-x\right)}dx = \int_0^{\pi/4} 1$~~

6. $I = \int_0^{\pi/4} \frac{\cos x-\sin x}{\cos x+\sin x}dx = \Big[\log[\cos x+\sin x]\Big]_0^{\pi/4}$

$$= \log\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right) - \log(1+0)$$

$$= \log\sqrt{2} \quad \underline{\underline{\text{Ans}}}.$$

Results :—

(1) . $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\,\frac{n-3}{n-2}$ ---- {upto 0 or -ve term}

$*\frac{\pi}{2}$; if n is even, Multiply with $\frac{\pi}{2}$

Ex :– $\int_0^{\pi/2} \sin^5 x\,dx = \frac{4}{5}\cdot\frac{2}{3}$

- $\int_0^{\pi} \sin^2 x\,dx$ Rule ⑥

$\sin^2(\pi-x) = \sin^2 x$

$= 2\int_0^{\pi/2} \sin^2 x\,dx = 2\cdot\frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{2}$

- $\int_0^{\pi/2} \cos^6 x\,dx = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\times\frac{\pi}{2}$

(2) $\int_0^{\pi/2} \sin^n x\cos^m x\,dx = \frac{(n-1)(n-3)\text{------}(m-1)(m-3)\text{----}}{(m+n)(m+n-2)\text{------}}$ $\left\{\times\frac{\pi}{2}\right.$ if m & n both are even.

0 $\quad 8\cdot6\cdot4\cdot2 \quad 24 \quad 0 \quad 9\cdot7\cdot5\cdot\cancel{3} \quad 315$

(3) $$\int_0^{\pi/2} \sin^6 x \cos^4 x\,dx = \frac{\cancel{5}\cdot\cancel{3}\cdot 1\cdot 3\cdot 1}{\cancel{10}_2\times 8\times \cancel{6}_2\times 4\times 2}\times\frac{\pi}{2} = \frac{3\pi}{512}$$

__Ex :—__ 1) $\int_0^1 \frac{x^6}{\sqrt{1-x^2}}dx$ Put $x=\sin\theta$, $dx=\cos\theta\,d\theta$; $x=0\rightarrow\theta=0$, $x=1\rightarrow\theta=\pi/2$

$$\int_0^{\pi/2}\frac{\sin^6\theta}{\cancel{\sqrt{1-\sin^2\theta}}}\cancel{\cos\theta}\,d\theta = \int_0^{\pi/2}\sin^6\theta\,d\theta = \frac{\cancel{5}}{\cancel{6}_2}\cdot\frac{\cancel{3}}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}$$

2) $I=\int_0^{\pi} x\sin^6 x\cos^4 x\,dx = \int_0^{\pi} x\sin^6 x\cos^4 x\,dx$ Rule ② $I=\int_0^{\pi}(\pi-x)\sin^6\cos^4 dx$

$$2I = \pi\int_0^{\pi}\sin^6 x\cos^4 x\,dx = ~~\frac{\pi\, 5\cdot3\cdot3}{2\,10\cdot8\cdot6\cdot4\cdot2}~~$$

$$2I = 2\pi\int_0^{\pi/2}\sin^6 x\cos^4 x\,dx = \frac{\cancel{5}\cdot\cancel{3}\cdot 1\cdot 3}{\cancel{10}_2\cdot8\cdot\cancel{6}_2\cdot4\cdot2}\times\frac{\pi}{\cancel{2}}\times\cancel{2}\pi = \frac{3\pi^2}{512}$$ __Ans.__

__Limiting Case Prblm :—__

(1) $$\lim_{n\to\infty}\left[\frac{n}{n^2}+\frac{n}{n^2+1^2}+ - - - + \frac{n}{n^2+(n-1)^2}\right] = \lim_{n\to\infty}\left[\sum_{K=0}^{n-1}\frac{n}{n^2+(k)^2}\right]$$

As $n\to\infty$, the summation can be approximated with integration (∴ limiting case of summation is integration).

$$= \lim_{n\to\infty}\int_{x=0}^{n-1}\frac{n}{n^2+x^2}dx = \lim_{n\to\infty} n\cdot\frac{1}{n}\tan^{-1}\frac{x}{n}\Big|_0^{n-1} = \lim_{n\to\infty}\tan^{-1}\left(\frac{n-1}{n}\right) - \tan^{-1}$$

$$= \lim_{n\to\infty}\tan^{-1}\left(1-\frac{1}{n}\right) = \tan^{-1}1 = \frac{\pi}{4}$$

(2) $\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}$ $\quad Y=\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}$ Apply log on both sides.

$$\log Y = \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{n!}{n^n}\right) = \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{1\cdot2\cdot3 ...... n}{nnn ---- n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\log\left(\frac{1}{n}\right)+\log\left(\frac{2}{n}\right)+ --- \log\left(\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{K=1}^{n}\log\left(\frac{K}{n}\right)$$ Approx. with int.

$$= \lim_{n\to\infty}\frac{1}{n}\int_{x=1}^{n}\log\left(\frac{x}{n}\right)dx$$ Put $\frac{x}{n}=t$, $dx=n\,dt$ Limits : $x=1$ $t=1/n$; $x=n$ $t=1$

$n \to \infty$ $\int_{1/n}$ $= \int_0 \log t \, dt$

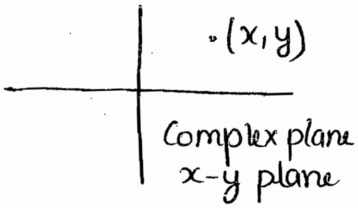
$$= t\log t - t\Big]_0^1 = -1 \quad \Rightarrow \log y = -1 \text{ or } y = \frac{1}{e} \text{ Ans.}$$

(3) $\underset{a \to 0}{\text{Lt}} \dfrac{x^a - 1}{a}$ $\qquad (a^x)' = a^x \log a$

$$\underset{a\to 0}{\text{Lt}} \frac{\frac{d}{da}(x^a - 1)}{\frac{d}{da} a} = \underset{a \to 0}{\text{Lt}} \frac{x^a \log x}{1} = \log x$$

Complex Analysis

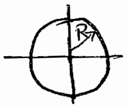
$z = x + iy$



(1) $\bar{z} = x - iy$

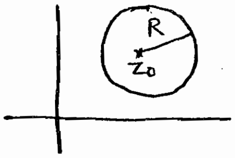
(2) $|z| = \sqrt{x^2+y^2}$

(3) $z \cdot \bar{z} = |z|^2$



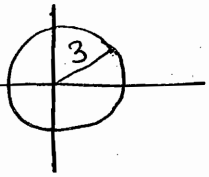
(4) $|z| = R$; Circle with centre at (0,0) & rad. R

(5) $|z - z_0| = R$; Circle with centre at z_0 & rad. R.



Ex:- C: $|z| = 3$

$2+3i$ lies outside C



Distance (0,0) to (2,3)

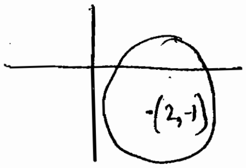
$= \sqrt{(2-0)^2 + (3-0)^2} = \sqrt{13} > 3$

Ex :- C: $|z - 2 + i| = 2$; Centre (2,−1) ; rad = 2

$1+i$ lies ____ C

Distance (2,−1) to (1,1) $= \sqrt{(1-2)^2 + (1+1)^2}$

$= \sqrt{1+4} = \sqrt{5} > 2$ lies outside C.



(6) Polar & Exponential Forms :

Cartesian form : $z = x + iy$

Polar form : $z = r[\cos\theta + i\sin\theta]$

exponential form ; $z = re^{i\theta}$

where r = Mod z

θ = arg z or amplitude z

$$\boxed{r = |z| = \sqrt{x^2+y^2} \;;\; \theta = \tan^{-1}\frac{y}{x}}$$

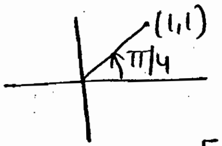
(7) Principle Argument : Arg z

$-\pi < \theta \le \pi$

$$\boxed{\underset{\underset{\text{Any argument}}{\downarrow}}{\arg z} = \underset{\underset{\text{principle argument}}{\downarrow}}{\text{Arg}\, z} + 2k\pi} \quad k = 0, \pm1, \pm2 ----$$

Ex :- Express $1+i$ in polar & exponential form : $z = 1+i$

$r = |z| = \sqrt{1^2+1^2} = \sqrt{2}$; $\theta = \tan^{-1}\frac{y}{x} = \underset{\underset{\text{Arg } z}{\uparrow}}{\frac{\pi}{4}}$



Polar $= \sqrt{2}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = 1+i$

exponential $=$ $z = \sqrt{2}e^{i\pi/4}$.

or

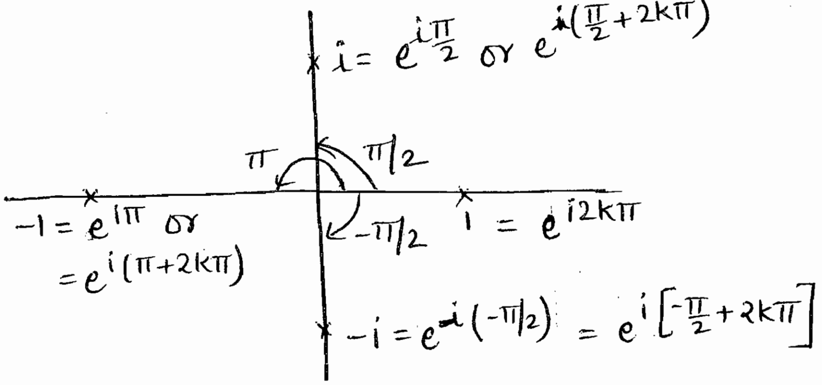
$z = \sqrt{2}\, e^{i\left(\frac{\pi}{4} + 2k\pi\right)}$

$K = 0, \pm1, \pm2 ----$

$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$, $\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1} 1 = -\frac{3\pi}{4}$

$z = \sqrt{2}\, e^{-i 3\pi/4}$ or $z = \sqrt{2}\, e^{+i\left[-\frac{3\pi}{4} + 2k\pi\right]}$

Remember –



Q:– $(i)^i$

$i = e^{i\pi/2}$

$(i)^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{-\pi/2}$

Q:– $\ln\sqrt{i} = \ln(i)^{1/2}$

$= \ln\left(e^{i\pi/2}\right)^{1/2}$

$= \cancel{\ln e}^{\,i\pi/4} = i\frac{\pi}{4}$

(8) n^{th} root of a complex no. z_0 ; $z_0 = r_0 e^{i\theta_0}$

The n distinct n^{th} roots of z_0 are given

$$\boxed{(z_0)^{1/n} = (r_0)^{1/n}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)}} \quad k = 0, 1, 2 \text{ ----- } n-1.$$

Ex:– Find square roots of i , find $(i)^{1/2}$

Solⁿ:– $i = e^{i\left(\frac{\pi}{2}+2k\pi\right)} \Rightarrow (i)^{1/2} = e^{i\left(\frac{\pi}{4} + \frac{\cancel{2}\pi k}{\cancel{2}}\right)} = e^{i\left(\frac{\pi}{4}+k\pi\right)}$

$k = 0, 1$

Distinct roots are :

$k = 0$; $e^{\frac{i\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}}$

$k = 1$; $e^{i\left(\frac{\pi}{4}+\pi\right)} = e^{i5\pi/4} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{-1-i}{\sqrt{2}}$

* Cube roots of unity $\Rightarrow 1, \omega, \omega^2$.

Ex:– $(x-1)^3 + 8 = 0$. Find roots of the eqⁿ.

Solⁿ:– $(x-1)^3 = -2^3$

$x - 1 = -2$

$x = -1$

$(x-1)^3 = (-2\omega)^3$

$x - 1 = -2\omega$

$x = 1 + 2\omega$

$(x-1)^3 = (-2\omega^2)^3$

$x - 1 = -2\omega^2$

$x = 1 - 2\omega^2$

Complex Function :— If $z = x + iy$ is complex variable.

$f(z) = u(x,y) + i\,v(x,y)$ is a complex fⁿ.

Ex:– $f(z) = z^2 = (x+iy)(x+iy)$

$f(z) = \underbrace{x^2 - y^2}_{\substack{\text{Real} \\ \text{"U"}}} + \underbrace{i2xy}_{\substack{\text{Imag.} \\ \text{"V"}}}$

$\left[z^2 \neq |z|^2\right]$

Let $f(z) = u+iv$. $f'(z)$ exist at z_0 iff the 1st partial derivatives u_x, u_y, v_x & v_y exist, continuous at z_0 & satisfy C-R equations.

$$\boxed{U_x = V_y \quad \& \quad U_y = -V_x}$$ at z_0

Also, $\boxed{f'(z) = U_x + iV_x}$ at z_0

Def :- Neighbourhood of z_0 :- nbd of z_0 ; $|z-z_0| < \epsilon$

Def :- $f(z)$ is analytic at z_0 if f is differentiable at every pt. in some nbd of z_0.

Def :- A pt. "z_0" where fⁿ $f(z)$ fails to be analytic is singular pt. or singularity of $f(z)$

Def :- A fⁿ $f(z)$ which is analytic throughout complex plane is called $\boxed{\text{entire } f^n}$

Milne Thompson Method - If $f(z) = u(x,y) + iv(x,y)$ is analytic fⁿ, then Put $\boxed{x=z \ \& \ y=0}$ in $f(z)$, $f'(z)$ ... to express in terms of z.

Ex :- $f(z) = (x^2-y^2) + i2xy$

a) analytic for all (x,y)

b) -||- for ~~all~~ $y=x$

c) -||- for $x>0, y>0$

d) None.

$U = x^2-y^2$, $V = 2xy$

$U_x = 2x$, $V_x = 2y$

$U_y = -2y$, $V_y = 2x$

$\Rightarrow U_x = V_y$ & $U_y = -V_x$

→ CR eqⁿ satisfies for all (x,y)

⇒ f is differentiable for all z.

⇒ f is analytic for all z.

⇒ entire fⁿ.

$f'(z) = U_x + iV_x$

$f'(z) = 2x + i2y$

∴ $f(z)$ analytic

Use Milne thompson method

$f(z) = z^2$, $f'(z) = 2z$

a) analytic $\forall (x,y)$ ~~b)~~** for $y=x$

c) nowhere analytic d) None

$U = x^2$, $V = y^2$, $U_x = 2x$, $V_y = 2y$

$U_y = 0$, $V_x = 0 \Rightarrow$ CR eqn satisfies only when $x = y$

$\Rightarrow f$ is diff. on $y = x$, but have to be diff. on surrounding

$\therefore$ No where analytic

Formula :—

If $f(z) = u + iv$ is analytic

1) $f'(z) = U_x + iV_x$

$\boxed{2)\ f'(z) = U_x - iU_y}$

3) $f'(z) = V_y - iU_y$

$\boxed{4)\ f'(z) = V_y + iV_x}$

Ex :— $u = x + y + xy$

$u_x = 1 + y \Rightarrow u = \underset{y\,const}{\int u_x\,dx} = \underset{y\,const}{\int (1+y)dx} = \boxed{x + xy + c_1}$

$u_y = 1 + x$

$\Rightarrow u = \underset{x\,const}{\int u_y\,dy} = \boxed{y + xy + c_2}$

$\Rightarrow$ Combining the two :—(comparing)

$\boxed{u = x + xy + y + c}$

Ex :— Let $f(z) = u + iv$ be analytic fn $u = e^x \cos y$. (1) Find V (2) find $f(z)$

Soln :— Find V (Method 1) :

$u = e^x\cos y$; $U_x = e^x \cos y = V_y$; $u_y = -e^x \sin y = -V_x$

(By CR eqns)

$V_x = e^x \sin y$; $V_y = e^x \cos y$

$V = \underset{y\,const}{\int e^x \sin y\,dx} = e^x \sin y + C_1$ $\quad$ $V = \underset{x\,const}{\int V_y\,dy} = \underset{x\,const}{\int e^x \cos y\,dy} = e^x \sin y + C_2$

$\therefore \boxed{V = e^x \sin y + C}$

Put $x = z$ & $y = 0$ [Milne Thompson]

$\boxed{f(z) = e^z + iC}$ or $e^z + C$

Method 2 :— find $f(z)$

$U = e^x \cos y$; $U_x = e^x \cos y$; $U_y = -e^x \sin$

$f'(z) = U_x - iU_y$

$f'(z) = e^x \cos y + ie^x \sin y$

M-T method : put $x = z$ & $y = 0$

$\int f'(z) = \int e^z \Rightarrow \boxed{f(z) = e^z + C}$

Ex :— $U = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$, $f(z) = U + iV$ is analytic

Find $f(z)$.

Soln :— $U_x = \dfrac{(\cosh 2y + \cos 2x)2\cos 2x - (P)}{(\cosh 2y + \cos 2x)^2}$

$P = \sin 2x\ 2(-\sin 2x)$

Put $x = z$ & $y = 0$

$$U_x = \frac{(1+\cos 2z)2\cos 2z + 2\sin^2 2z}{(1+\cos 2z)^2}$$

$$= \frac{2\cos 2z + 2(\cos^2 2z + \sin^2 2z)}{(1+\cos 2z)^2}$$

$$= \frac{2(\cancel{1+\cos 2z})}{(1+\cos 2z)^{\cancel{2}}} = \frac{2}{1+\cos 2z}$$

$$U_y = \frac{-\sin 2x}{(\cosh 2y + \cos 2x)^2} \cdot 2\sinh 2y$$

$x = z$ & $y = 0$

$U_y = 0$

$$f'(z) = U_x - iU_y = \frac{2}{1+\cos 2z}$$

$$f(z) = \int \frac{2}{2\cos^2 z}dz + c$$

$$= \int \sec^2 z\,dz + c$$

$\boxed{f(z) = \tan z + c}$

$u-v = (x-y)(x^2+4xy+y^2)$

Sol^n:— $u-v = x^3-y^3+3x^2y-3xy^2$

$if(z) = iu - v$ —②

①+② : $(1+i)f(z) = (u-v)+i(u+v)$

Let $(1+i)f(z) = F(z) = U + iV$

$U = x^3-y^3+3x^2y-3xy^2$

$U_x = 3x^2+6xy-3y^2$ Put $x=z, y=0$

$U_x = 3z^2$

$U_y = -3y^2+3x^2-6xy$ Put $x=z, y=0$

$U_y = 3z^2$

$F'(z) = U_x - iU_y = 3z^2 - i3z^2$

$F'(z) = (1-i)3z^2$

$$\boxed{F(z) = (1-i)z^3 + c}$$

$$\cancel{(1+i)}\, f(z) = \frac{1-i}{1+i} z^3 + \frac{c}{1+i}$$

$$f(z) = -\frac{\cancel{2}i}{\cancel{2}} z^3 + C$$

$\boxed{f(z) = -iz^3+C}$ Ans.

Result :— If $f(z) = U+iV$ is analytic then ① u & v are harmonic f^{n}s.

i.e. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

② V is called harmonic conjugate of u.

Taylor's Series —(T.S)

If $f(z)$ is analytic at $z=a$, then $f(z)$ can be expressed in ~~terms~~ powers of $(z-a)$ & this power series is called T.S. of $f(z)$ abt $z=a$ (or) T.S. of $f(z)$

$$\boxed{f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \text{---}}$$

If $a=0$, it is Mclaurin's Series

$$\boxed{f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \text{----}}$$

Ex :— 1) $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \text{----}, |z|<\infty$

2) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \text{----}, |z|<\infty$

3) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \text{----}, |z|<\infty$

4) $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \text{----} ; |z|<1$

Laurent's Series :—

Power Series of $f(z)$ in positive & negative powers of $(z-a)$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=}^{\infty} \frac{b_n}{(z-a)^n}$$

Ex :— Expand $f(z) = \frac{1}{(z-2)(z-1)}$ in powers of z.

(i) $|z|<1$ (ii) $1<|z|<2$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

(i) $|z|<1$; $\left|\frac{z}{2}\right|<1$

$$\frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{-2(1-z/2)} - \frac{1}{-1(1-z)}$$

$$= \frac{-1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\text{----}\right] + \left[1+z+z^2+\text{---}\right]$$

(ii) $1<|z|<2$

$|z|<2 \Rightarrow \left|\frac{z}{2}\right|<1$; $1<|z| \Rightarrow \left|\frac{1}{z}\right|<1$

$$\frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{-2(1-\frac{z}{2})} - \frac{1}{z(1-\frac{1}{z})}$$

$$= \frac{-1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\text{----}\right] - \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\text{---}\right]$$

(1) Polynomial f^n : $P(z)$

→ entire f^n.

→ Zeroes : $p(z) = 0$

***(2) Rational f^n : $R(z)$

$$R(z) = \frac{P(z)}{Q(z)}$$

*** → having singularities, $Q(z) = 0$

→ Zeroes : $P(z) = 0$

Ex:- $\frac{z^3+3z}{(z-2)(z-3)}$ Singularities → 2, 3

$\frac{z^2+3z+5}{z^2+1}$ $z^2+1 = 0$; $z = +i, -i$

(3) e^z

→ $e^z = e^x\cos y + ie^x\sin y$

→ entire f^n

→ $e^z \neq 0$

→ periodicity → $\boxed{2\pi i}$

(4) $\sin z \cos z$

→ entire f^ns

**** → Zeroes :

$\sin z = 0$ iff $z = n\pi$; $n = 0, \pm1, \pm2, ---$

$\cos z = 0$ iff $z = (2n+1)\frac{\pi}{2}$; $n = 0, \pm1, \pm2$

→ periodicity :

$\sin z, \cos z, \sec z, \mathrm{cosec}\, z$ → 2π

$\tan z$, $\cot z$ → π

(5) $\sinh z \cosh z$

→ entire f^n

→ Zeroes:

$\sinh z = 0$ iff $z = n\pi i$; $n = 0, \pm1, \pm2 ...$

$\cosh z = 0$ iff $z = (2n+1)\frac{\pi}{2}i$; $n = 0, \pm1, \pm2 ...$

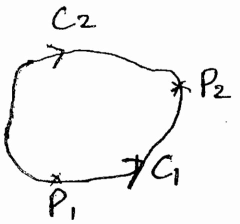
→ Periodicity

→ $2\pi i$

→ πi

(Contour Integration)-

$$\int_C f(z)dz$$



In general,

$$\int_{P_1}^{P_2} f(z)dz \neq \int_{P_1}^{P_2} f(z)dz$$

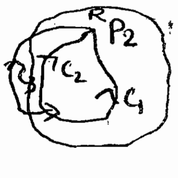
along C_1 along C_2

Complex integration depends on path of integration.

Result - If $f(z)$ is analytic f^n in a region R then

$$\int_{P_1}^{P_2} f(z)dz = \int_{P_1}^{P_2} f(z)dz$$

along C_1 along C_2



where C_1 & C_2 lies in R.

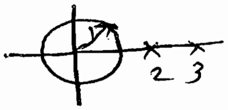
Cauchy Integral Theorem -

If $f(z)$ is analytic inside & on Closed curve C then $\oint_C f(z)dz = 0$

Ex:- $\oint_C \frac{z^2+3z+4}{(z-2)(z-3)}dz$ C is circle $|z| = 1$

Sol^n:- Singularities, $z = 2, 3$ both lies outside C.

By CIT, $\oint_C = 0$



(2) Singularities of $\tan z$

$$\tan z = \frac{\sin z}{\cos z}$$

Singularities : $\cos z = 0$ iff

$\boxed{z = (2n+1)\frac{\pi}{2}}$ $n = 0, \pm1, \pm2$

Singularities : $z = (2n+1)\pi/2$

$n = 0, \pm 1, \pm 2 \_\_\_\_$

$\_\_\_\_ \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \_\_\_\_$

$\_\_\_\_ -1.57, 1.57 \_\_\_\_$

all lies outside C $\therefore \oint_{c:|z|=1} \tan z\, dz = 0$

Cauchy Integral Formula –

If f(z) is analytic inside & on c & "a" is any point inside "C".

$$\boxed{f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}\,dz}$$

$$\Rightarrow \oint \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

Result :–

1) $f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}\,dz$

2) $f'(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^2}\,dz$

3) $f''(a) = \frac{2!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^3}\,dz$

4) $f'''(a) = \frac{3!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^4}\,dz$

$\vdots$

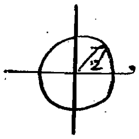
5) $f^{(n)}(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}\,dz$

6) $\oint \frac{f(z)}{(z-a)^n} = \frac{2\pi i}{(n-1)!} f^{n-1}(a)$

Q:– $_c\oint \frac{z^2+3z+2}{(z-1)(z-3)}\,dz \quad c:|z| = 2$

Soln :– Singularities 1, 3

Only z=1 lies inside c.



$_c\oint \frac{(\;z-3\;)}{(z-1)}\,dz = 2\pi i f(1)$

$= 2\pi i \times (-3) = -6\pi i$

$f(1) = \frac{6}{-2} = -3$

Q:– $\oint \frac{z^2+3z+2}{(z-1)^2(z-3)}\,dz \quad c: |z| = 2$

Singularities : 1, 3

Only z=1 lies inside C.

$_c\oint \frac{\frac{z^2+3z+2}{z-3}}{(z-1)^2}\,dz = 2\pi i f'(1)$

$= \boxed{-8\pi i}$

$f'(z) = \frac{(z-3)(2z+3) - (z^2+3z+2)}{(z-3)^2}$

$f'(1) = \frac{-2(5)-6}{4} = \frac{-16}{4} = -4$

Q:– $_c\oint (z-a)^n dz \quad c: |z|>1$ & $a>1$

Soln :- By Cauchy I.T :

$_c\oint (z-a)^n dz = 0$ ∵ there are no sing.

Q:– $_c\oint \frac{1}{z-a}dz \quad c: |z| = 1 \quad a<1$ (a is inside C)

By C.I.F :

$_c\oint \frac{1}{z-a}dz = 2\pi i f(a) = \boxed{2\pi i}$

$f(z) = 1, \quad f(a) = 1$

Q:- $_c\oint \frac{1}{(z-a)^n}dz$; $c:|z| = 1$; a is inside C & $n>1$

$_c\oint \frac{1}{(z-a)^n}dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(a) = 0$

$f(z) = 1, \quad f(a) = 1$

$f'(z) = 0, \quad f^{n-1}(z) = 0$

Isolated singulality –

There is no other singularity within some nbd of singularity a, then a is isolated singularity.

Laurent's Theorem :–

If "a" is isolated singularity of f(z) then f(z) can be expressed as L.S (Laurent's series) abt $z=a$ (L.S with centre at $z=a$)

$$f(z)=\underbrace{\sum_{n=0}^{\infty}a_n(z-a)^n}_{\text{Analytic Part}}+\underbrace{\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}}_{\text{Principle Part}}$$

Def – If the principle part of L.S. representation of f(z) about singularity "a" contains

(i) No terms → "a" is called removable singularity.

(ii) Infinite terms → "a" is called essential singularity.

(iii) finite terms (say m) → "a" is called pole of order m.

Residue of f(z) at isolated singularity "a"

Residue $= b_1 =$ coefficient of $\frac{1}{z-a}$

= coefficient of 1st –ve power of (z-a) in L.S. representation of f(z) abt $z=a$.

Result –

Let $f(z)=\frac{g(z)}{(z-a)^m}$ then

f(z) has pole of order m at $z=a$ if g(z) is analytic at a & $g(a)\neq 0$

Q:– $f(z)=\frac{z^3+3z+4}{(z-2)^2(z-3)^5}$

Singularities : 2, 3

$z=2$: pole of order 2

Residue of f(z) at "a" where "a" is pole of order m,

$$\boxed{\operatorname*{Res}_{z=a} f(z)=\lim_{z\to a}\left[\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]\right]}$$

pole of order 2 → double pole.
pole of order 1 → simple pole.

Residue at simple pole $z=a$

$$\operatorname*{Res}_{z=a} f(z)=\lim_{z\to a}\left[(z-a)f(z)\right]$$

Residue at double pole $z=a$

$$\operatorname*{Res}_{z=a} f(z)=\lim_{z\to a}\left[\frac{d}{dz}\left[(z-a)^2 f(z)\right]\right]$$

Q:– Find residue of f(z) at singularities

(1) $f(z)=\frac{z^3+5z^2+6z+5}{(z-3)^3}$

$z=3$ is pole of order 3.

$$\operatorname*{Res}_{z=3} f(z)=\lim_{z\to 3}\left[\frac{1}{(3-1)!}\frac{d^2}{dz^2}\left[(z-3)^3\,\frac{z^3+5z^2+6z+5}{(z-3)^3}\right]\right]$$

$$\Rightarrow \lim_{z\to 3}\left[\frac{1}{2}\frac{d}{dz}\left[3z^2+10z+6\right]\right]$$

$$\Rightarrow \lim_{z\to 3}\left[\frac{1}{2}\left[6z+10\right]\right]=\frac{28}{2}=14$$

Q:– Find residues of f(z) at singularities

$f(z)=\frac{z^2+6z+5}{(z-1)^2(z-2)}$

$z=1$ is pole of order 2 (double pole).
$z=2$ is simple pole.

$$R_1=\operatorname*{Res}_{z=1} f(z)=\lim_{z\to 1}\left[\frac{d}{dz}\left[(z-1)^2\,\frac{z^2+6z+5}{(z-1)^2(z-2)}\right]\right]$$

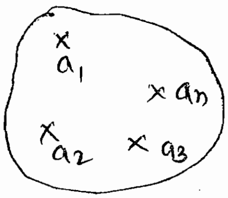
$$=\lim_{z\to 1}\frac{(z-2)(2z+6)-(z^2+6z+5)\cdot 1}{(z-2)^2}$$

$$=-8-12=-20$$

Let $f(z)$ be analytic inside & on c except for finite no. of isolated singularities $a_1, a_2 ---, a_n$ inside C.

Let $R_1, R_2 ----- R_n$ be residues at $a_1, a_2, ---- a_n$ respectively.

$$\boxed{{}_c\oint f(z)dz = 2\pi i\left[R_1+R_2+-----+R_n\right]}$$



Q:- $_c\oint \frac{z^2+6z+5}{(z-1)^2(z-2)}dz$; $c: |z|=3$

Singularities → 1, 2 both lies Inside c.

$z=1$ is double pole ;
$z=2$ is simple pole.

$R_1 = -20$; $R_2 = 21$

∴ By Residue Theorem :

$$_c\oint \frac{z^2+6z+5}{(z-1)^2(z-2)}dz = 2\pi i\left[R_1+R_2\right]$$
$$= 2\pi i\left[21-20\right]$$
$$= 2\pi i$$

Q:- $_c\oint \tan z\, dz$ $C: |z|=2$

Singularities : $\cos z = 0 \Leftrightarrow z = (2n+1)\frac{\pi}{2}$

$n = 0, \pm 1, \pm 2 -----$

$\Rightarrow ---- -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} ---------$

$z = -\frac{\pi}{2}$ & $\frac{\pi}{2}$ lies inside c

$z = \frac{\pi}{2}$ is a simple pole.

$z = -\frac{\pi}{2}$ is a simple pole.

$$R_{\pi/2} = \underset{z\to\frac{\pi}{2}}{Lt}\left(z-\frac{\pi}{2}\right).\tan z$$
$$= \underset{z\to\pi/2}{Lt}\frac{(z-\pi/2)\sin z}{\cos z}$$
$$= \underset{z\to\frac{\pi}{2}}{Lt}\frac{(z-\pi/2)\cos z+\sin z}{-\sin z} = \frac{1}{-1} = -1$$

$$R_{-\pi/2} = \underset{z\to-\pi/2}{Lt}\frac{(z+\pi/2)\sin z}{\cos z} = \cancel{\underset{z\to\frac{-\pi}{2}}{Lt}\left(z+\frac{\pi}{2}\right)}$$
$$= \underset{z\to\frac{-\pi}{2}}{Lt}\frac{(z+\pi/2)\cos z+\sin z}{-\sin z} = -1$$

$$_c\oint \tan z\, dz = 2\pi i\left[-1-1\right] = -4\pi i$$

Q:- $_c\oint \frac{e^{2z}}{z^2+1}dz$ $c: |z|=3$

Singularities : $z^2+1=0$; $z=\pm i$

$z=\pm i$ lies inside C.

$z=i$ simple pole, $z=-i$ simple pole.

$$R_i = \underset{z\to i}{Lt}\frac{\cancel{(z-i)}e^{2z}}{\cancel{(z-i)}(z+i)} = \frac{e^{2i}}{2i}$$

$$R_i = \underset{z\to -i}{Lt}\frac{(z+i)e^{2z}}{(z+i)(z-i)} = \frac{e^{-2i}}{-2i}$$

$$_c\oint \frac{e^{2z}}{z^2+1}dz = 2\pi i\left[R_i+R_{-i}\right]$$
$$= 2\pi i\left[\frac{e^{2i}-e^{-2i}}{2i}\right]$$
$$= 2\pi i(\sin 2)$$

Q:- $f(z) = \frac{\sin z}{z^5}$ has ______ at $z=0$

a) pole of order 3 ~~b)~~ 4 c) 5 d) 6

$$f(z) = \frac{1}{z^5}\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+-----\right]$$
$$= \textcircled{\frac{1}{z^4}} - \frac{1}{z^2 3!} + \frac{1}{5!} - \frac{z^2}{7!} + ----$$

$\frac{\sin z}{z^5}$ has pole of order 4 at $z=0$ as highest −ve power is ~~po~~ 4.

Residue of $\frac{\sin z}{z^5}$ at $z=0$ is 0

∵ coeff. of $\frac{1}{z}$ is 0.

$$\frac{1}{z^4} + \frac{1}{z^3}0 - \frac{1}{z^2 3!} + \frac{1}{\cancel{2}\,z}\textcircled{0} + \frac{1}{5!}$$

~~2term~~

a) Removable Singularity b) Essential Sing.

c) Simple pole. d) Double pole.

Solⁿ:-

$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{z^3 3!} + \frac{1}{z^5 5!} - ----$

Residue is 1 ∵ coeff. of $\frac{1}{z}$ is 1.

As max^m power is ∞.

VECTOR CALCULUS

Gradient → Scalar fⁿs to vector fⁿs.

Divergence → Vector fⁿs to scalar fⁿs.

Curl → Vector fⁿs to vector fⁿs.

Del or Nabla –

$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

Gradient –

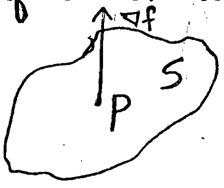
grad $f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = \nabla f$

1) Unit Normal :–

Let $f(x,y,z) = c$ denote surface.

∇f at P denotes surface normal vector at P.

$\boxed{\hat{N} = \frac{\nabla f}{|\nabla f|}}$



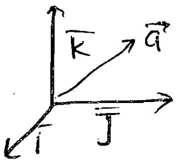
2) Angle b/w two surfaces $f(x,y,z) = c_1$ & $g(x,y,z) = c_2$ at P

$\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|}$ at P

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

direcⁿ of –

1. $\bar{i}$ (x-axis) is $\partial f/\partial x$
2. $\bar{j}$ (y-axis) is $\partial f/\partial y$
3. $\bar{k}$ (z-axis) is $\partial f/\partial z$



(4) D.D. of f at P in direction $\vec{a}$

$D.D_{\vec{a}} f = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$ at P.

(5) D.D. of f will be max in direcⁿ ∇f. $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ max at $\theta = 0°$.

(6) Max. value of $D.D = |\nabla f|$

$D.D_{\nabla f} f = \nabla f \cdot \frac{\nabla f}{|\nabla f|}$

$= \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|$

Q:– Find unit normal to the surface $f = 3x^2y - y^3z^2$ at $(1,-2,-1)$

Solⁿ:– $\hat{N} = \frac{\nabla f}{|\nabla f|}$

$\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$

$= (6xy - 0)\bar{i} + (3x^2 - 3y^2z^2)\bar{j} + (0 - 2y^3z)\bar{k}$

$\nabla f_{(1,-2,-1)} = -12\hat{i}$ [illegible] $9\hat{j} - 16\hat{k}$.

$|\nabla f| = \sqrt{(-12)^2 + (-9)^2 + (-16)^2} = \sqrt{481}$

$\hat{N} = \frac{-12\bar{i} - 9\bar{j} - 16\bar{k}}{\sqrt{481}}$

Q:– Find unit normal to the sphere $x^2 + y^2 + z^2 = 4$

$f: x^2 + y^2 + z^2 = 4$

$\nabla f = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

$|\nabla f| \quad \sqrt{4x^2+4y^2+4z^2}$

$$= \frac{\cancel{2}[x\hat{i} + y\hat{j} + z\hat{k}]}{\cancel{2}\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{4}}$$ on surface of sphere $x^2+y^2+z^2=4$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{2}$$

Q:- Find D.D. of $f = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direcⁿ of $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$

Solⁿ:- $D.D_{\vec{a}} f = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$ at P.

$\nabla f = [2xyz + 4z^2]\hat{i} + [x^2z + 0]\hat{j} + [x^2y + 8xz]\hat{k}$

$(\nabla f)_{(1,-2,-1)} = 8\bar{i} - \bar{j} - 10\bar{k}$

$\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$

$|\vec{a}| = \sqrt{4+1+4} = 3$

$D.D_{\vec{a}} f = (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \frac{(2\hat{i} - \hat{j} - 2\hat{k})}{3}$

$= \frac{16+1+20}{3} = \frac{37}{3}$

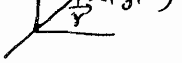
~~Q:- greatest value of D.D of f, $f = x^2yz + 4xz^2$ at $(1,-2,-1)$~~

Solⁿ:- $|\nabla f| = \sqrt{64+1+100} = \sqrt{165}$ is the greatest value or maxᵐ value of the fⁿ.

Position Vector of point (x,y,z):-

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

length $\vec{r} = |\vec{r}|$



Results -

(1) $\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}$

$= \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x = \frac{x}{r}$

$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$ (2) $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$

(3) $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

(4) $\nabla r = \frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}$

$= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$

$\boxed{\nabla r = \frac{\vec{r}}{r}}$

(5) $\nabla f(r) = \frac{\partial}{\partial x} f(r)\hat{i} + \frac{\partial}{\partial y} f(r)\hat{j} + \frac{\partial}{\partial z} f(r)\hat{k}$

$\frac{\partial}{\partial x} f(r) = f'(r)\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}$

$\nabla f(r) = f'(r)\frac{x}{r}\hat{i} + f'(r)\frac{y}{r}\hat{j} + f'(r)\frac{z}{r}\hat{k}$

$\boxed{\nabla f(r) = \frac{f'(r)}{r}[\vec{r}]}$

(6) $\nabla \log r = \frac{1}{r}\nabla r = \frac{1}{r}\frac{\vec{r}}{r} = \boxed{\frac{\vec{r}}{r^2}}$

(7) $\nabla\left(\frac{1}{r}\right) = \frac{-1}{r^2}\nabla r = \boxed{\frac{-\vec{r}}{r^3}}$

(8) $\nabla(r^n) = nr^{n-1}\cdot\nabla r$

$= nr^{n-1}\cdot\frac{\vec{r}}{r}$

$\boxed{\nabla(r^n) = nr^{n-2}\vec{r}}$

$$\text{Div } \vec{F} = \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Curl — Curl $F = \nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

Def :— $\nabla\cdot\vec{F} = 0$ then $\vec{F}$ is solenoidal vector.

$\nabla \times \vec{F} = \vec{0}$ then $\vec{F}$ is irrotational

Ex :— ① Find P, ; $\bar{F} = (x+3y)\hat{i} + (y+2z)\hat{j} + (x+pz)\hat{k}$ is solenoidal.

② $F = r^n \vec{r}$ is solenoidal then $n =$ ____

Sol^n :— ① $\nabla\cdot F = 0$; $\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y+2z) + \frac{\partial}{\partial z}(x+pz) = 0$

$$1+1+p=0 \Rightarrow p=-2$$

② $\vec{F} = r^n[x\hat{i} + y\hat{j} + z\hat{k}] = r^n x\hat{i} + r^n y\hat{j} + r^n z\hat{k}$

$\nabla\cdot\vec{F} = 0$

$\frac{\partial}{\partial x}(r^n x) = r^n + x.nr^{n-1}\frac{\partial r}{\partial x} = r^n + nxr^{n-1}\frac{x}{r}$

$\frac{\partial}{\partial x}(r^n x) = r^n + nx^2 r^{n-2}$

Similarly, $\frac{\partial}{\partial y}(r^n y) = r^n + ny^2 r^{n-2}$; $\frac{\partial}{\partial z}(r^n z) = r^n + nz^2 r^{n-2}$

$\frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z) = 3r^n + nr^{n-2}[x^2+y^2+z^2]$

$$\nabla\cdot F = 3r^n + nr^{n-2}r^2 = 0 \Rightarrow 3r^n + nr^n = 0$$

$$(3+n)r^n = 0 \Rightarrow n = -3$$

Q:— Find Curl of $F = yz\bar{i} + zx\bar{j} + xy\bar{k}$

Sol^n :— $\nabla\times\vec{F} = \begin{vmatrix} \bar{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \partial/\partial z \\ yz & zx & xy \end{vmatrix} = i[x-x] - \hat{j}[y-y] + \bar{k}[z-z] = \vec{0}$

$\vec{F}$ is irrotational.

Def :- F is conservative field if there exist a scalar f^n ϕ such that

$$\boxed{F = \nabla\phi}$$

F is irrotational.

$$\bar{F} = \nabla\phi \text{ iff } \nabla \times \bar{F} = 0$$

$$\phi \rightarrow \boxed{\text{scalar potential}}$$

Q:- Find scalar potential of

$$\bar{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

1) $\nabla \times \bar{F} = \vec{0}$

∴ $\bar{F}$ is conservative field.

$$\bar{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Partial Integration:

$$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &= yz \Rightarrow \phi = xyz + C_1 \\ \frac{\partial\phi}{\partial y} &= zx \Rightarrow \phi = xyz + C_2 \\ \frac{\partial\phi}{\partial z} &= xy \Rightarrow \phi = xyz + C_3 \end{aligned}\right\} \Rightarrow \boxed{\phi = xyz + C}$$

Line Integral -

$$\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C F_1 dx + F_2 dy + F_3 dz$$

Ex:- 1) $\bar{F} = 3xy\bar{i} - y^2\bar{j}$

Soln:- $\int_{C_1} \bar{F}\cdot d\bar{r}$ where C_1 is arc of parabola $y = 2x^2$ from (0,0) to (1,2)

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_{C_1} 3xy\,dx - y^2 dy$$

Parametric Form :-

$x = t$ & $y = 2t^2$

$dx = dt$ & $dy = 4t\,dt$

x varies from 0 to 1

x = t varies from 0 to 1

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} 3t(2t^2)dt - (4t^4)\cdot 4t\,dt$$

$$= \int_{t=0}^{1} [6t^3 - 16t^5]\,dt$$

$$= \left.\frac{6t^4}{4} - \frac{16t^6}{6}\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = -\frac{7}{6} \text{ Ans.}$$

② $\int_{C_2} \bar{F}\cdot d\bar{r}$ where C_2 is line joining pts from (0,0) to (1,2)

Remember -

Parametric form of line joining pts (x_1, y_1, z_1) to (x_2, y_2, z_2)

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

Soln:- Parametric form :

$$\frac{x - 0}{1 - 0} = \frac{y - 0}{2 - 0} = t \Rightarrow x = t, \ y = 2t$$

$dx = dt$ & $dy = 2dt$

(0,0) to (1,2)

x varies from 0 to 1

t = x varies from 0 to 1

$$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_{C_2} 3xy\,dx - y^2 dy$$

$$= \int_{t=0}^{1} 3t(2t)\,dt - 4t^2\, 2dt$$

$$= -2\left[\frac{t^3}{3}\right]_0^1$$

$$= -\frac{2}{3} \text{ Ans.}$$

1) In general,

$$\int_{P_1}^{P_2} \bar{F}\cdot d\bar{r} \neq \int_{P_1}^{P_2} \bar{F}\cdot d\bar{r}$$

along C_1 along C_2

2) If $\bar{F}$ is conservative field $(\bar{F} = \nabla\phi)$

then $\int_{P_1}^{P_2} \bar{F}\cdot d\bar{r} = \int_{P_1}^{P_2} d\phi = [\phi]_{P_1}^{P_2}$

along only C

Ex:- $\bar{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$

Find $\int_C \bar{F}\cdot d\bar{r}$ where C is the line joining point (1,1,1) to (1,2,3)

1) $\nabla \times \bar{F} = 0$ 2) $\bar{F} = \nabla\phi$

3) $\phi = xyz + c$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{(1,1,1)}^{(1,2,3)} d\phi = [\phi]_{(1,1,1)}^{(1,2,3)}$$

$$= xyz + c\Big]_{(1,1,1)}^{(1,2,3)}$$

$= 6 + c - 1 - c = 5$ Ans.

Multiple Integrals

Ex:- ① $\int_0^2 \int_0^1 xy\,dx\,dy = \int_0^2 \frac{x^2}{2}\Big|_0^1 y\,dy$
$y \quad x$

$$= \int_0^2 \frac{y}{2}\,dy = \frac{y^2}{4}\Big|_0^2 = 1$$

② $\int_{x=0}^1 \int_{y=0}^2 xy\,dy\,dx = \int_0^1 x\Big|\frac{y^2}{2}\Big|_0^2 dx$

$$\int_0^1 2x\,dx = x^2\Big|_0^1 = 1$$

③ $\int_0^1 x\,dx \int_0^2 y\,dy = \frac{x^2}{2}\Big|_0^1 \frac{y^2}{2}\Big|_0^2$

$$= \frac{1}{2}.2 = 1$$

We cannot separate in terms of x & y, so ③ is not applicable here.

$$\int_0^2 \frac{x^2}{2} + yx\Big|_0^1 dy = \int_0^2 \left(\frac{1}{2} + y\right)dy$$

$$= \frac{y}{2} + \frac{y^2}{2}\Big|_0^2 = \cancel{4}$$

$$= 2 + 1 = 3$$

Ex:- $\int_{x=0}^1 \int_{0=y}^x x^2y\,dy\,dx$

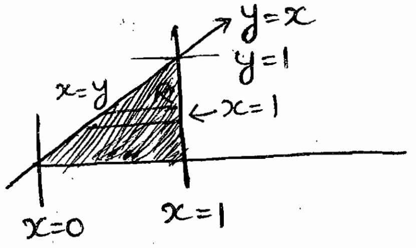
→ Double integration is carried out over a region.

$$\int_0^1 x^2 \frac{y^2}{\cancel{2}}\Big|_0^x dx = \int_0^1 \frac{x^4}{2} dx$$

$$= \frac{x^5}{10}\Big|_0^1 = \frac{1}{10}$$ Ans.

Change of order - variable limits decides shape

$y = 0$ to x , $x = 0$ to 1
const limit decides boundary



limits becomes : $x = y$ to 1 &
$y = 0$ to 1

$$\int_{x=0}^1 \int_{y=0}^x x^2y\,dy\,dx = \int_{y=0}^1 \int_{x=y}^1 x^2y\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3}{3}\right]_y^1 y\,dy = \int_0^1 \left[\frac{1}{3} - \frac{y^3}{3}\right] y\,dy$$

$$= \frac{1}{3}\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{1}{3}\left[\frac{5-2}{10}\right]$$

$$= \frac{1}{10}$$ Ans.

over a region.

- Triple integrals are carried out over volumes.

Formula -

Area of Region :

$$\boxed{A = \iint_R dxdy}$$

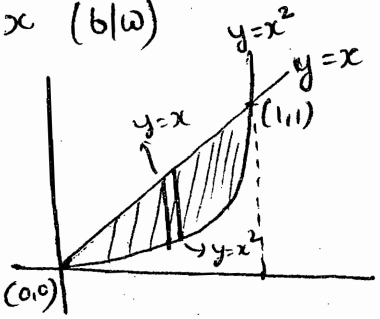
Note :- $\iint_R f(x,y)\,dxdy$ is not area of region.

Volume enclosed by closed surface :

$$\boxed{V = \iiint_V dxdydz}$$

Q:- Find the area of region bounded by $y=x^2$ & $y=x$ (b/w)

$$A = \iint_R dxdy$$



Limits :

$y=x \quad y=x^2 \Rightarrow x^2=x$

$x(x-1)=0$

$x=0,1$

$y=0,1$

$(0,0) \quad (1,1)$

Limits of Region :

$y=x^2$ to x

$x=0$ to 1

$$A = \int_0^1 \int_x^x dydx = \int_0^1 [x-x^2]dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6} \text{ Ans.}$$

$x=y$ to $x=\sqrt{y}$

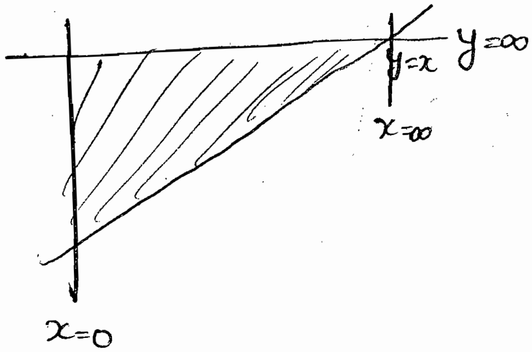
$y = 0$ to 1

$$A = \int_0^1 \int_y^{\sqrt{y}} dxdy = \int_0^1 [\sqrt{y}-y]dy$$

$$= \frac{1}{6} \text{ Ans.}$$

Ex :- $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dxdy$

$y = x$ to ∞ ; $x = 0$ to ∞



Change the order :-

Limits : $x=0$ to y & $y=0$ to ∞.

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dydx = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dxdy$$

$$= \int_0^\infty \frac{e^{-y}}{y} [x]_0^y dy = \int_0^\infty e^{-y}dy$$

$$= 0+1 = 1 \text{ Ans.}$$

Jacobian of the transformation :-

1) Let $u = f_1(x,y)$, $v = f_2(x,y)$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

2) Let $u = f_1(x,y,z)$,

$v = f_2(x,y,z)$

$w = f_3(x,y,z)$

$$\partial(x,y,z) \quad \begin{vmatrix} V_x & V_y & V_z \\ W_x & W_y & W_z \end{vmatrix}$$

Formulas for change of variables :-

Double Integral :

$$_R\iint f(u,v)\,du\,dv = \iint_R f(f_1,f_2)|J|\,dx\,dy.$$

Triple Integral :

$$_V\iiint f(u,v,w)\,du\,dv\,dw =$$

$$_V\iiint f(f_1,f_2,f_3)|J|\,dx\,dy\,dz.$$

Polar coordinates -

$x = r\cos\theta$, $y = r\sin\theta$

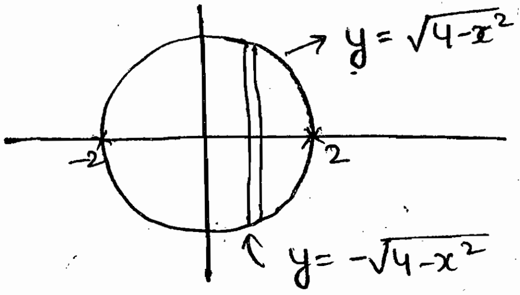
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\left[\cos^2\theta + \sin^2\theta\right] = r$$

Ex :-

$$_R\iint f(x,y)\,dx\,dy = {}_R\iint f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Ex :- Circle $x^2+y^2=4$



$R: x^2+y^2=4$

Limits : $y = -\sqrt{4-x^2}$ to $+\sqrt{4-x^2}$

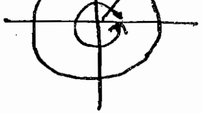
$x = -2$ to 2

Changing into polar coordinates :

$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$\theta = 0$ to 2π

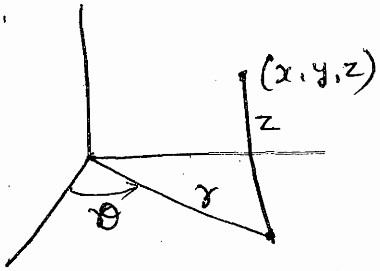


Cylindrical Polar Coordinates -

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$



$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

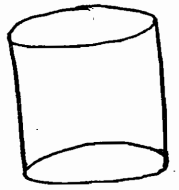
Result :- Cartesian to cylindrical polar coordinates.

$$\iiint_V f(x,y,z)\,dx\,dy\,dz =$$

$$_V\iiint f(r\cos\theta, r\sin\theta, z)\,\boxed{r\,dr\,d\theta\,dz}.$$

Ex :- Cylinder :

$x^2+y^2=4$, $z=0$ to $z=5$



Limits : $z=0$ to $z=5$

$y = -\sqrt{4-x^2}$ to $\sqrt{4-x^2}$

$x = -2$ to 2

x & y takes care of circle (i.e. boundary), z take care of shape in the space.

Using cylindrical polar :-

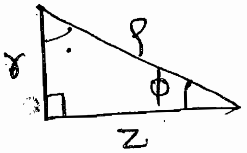
$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

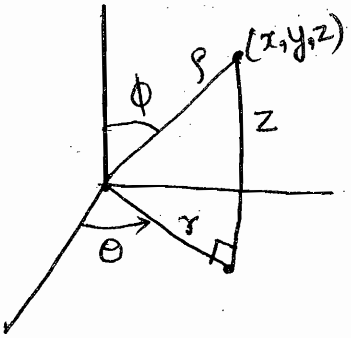
$$\boxed{dx\,dy\,dz = r\,dr\,d\theta\,dz}$$

Limits : $z=0$ to $z=5$

$r=0$ to 2

$\theta=0$ to 2π





$\sin\phi = \frac{r}{\rho}$ & $\cos\phi = z/\rho$

$r = \rho\sin\phi, z = \rho\cos\phi$

~~$x = r\cos\theta, y = r\sin\theta, z =$~~

$$\boxed{x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi}$$

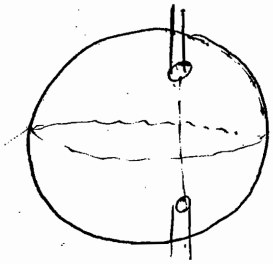
$$J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin\phi$$

Cartesian to spherical polar coordinates

$$\iiint_V f(x,y,z)\,dxdydz =$$

$$\iiint_V f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi \; d\rho d\phi d\theta$$

Ex :- Sphere $x^2+y^2+z^2=4$



$z = \mp\sqrt{4-x^2-y^2}$

$z = -\sqrt{4-x^2-y^2}$ to $+\sqrt{4-x^2-y^2}$.

$z \rightarrow$ takes care shape in the space.

$z=0, y = -\sqrt{4-x^2}$ to $+\sqrt{4-x^2}$

$z, y=0, x = -2$ to $+2$.

Limits of shere :—

Using spherical polar coordinates

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$$\boxed{dxdydz = \rho^2\sin\phi \, d\rho d\phi d\theta}$$

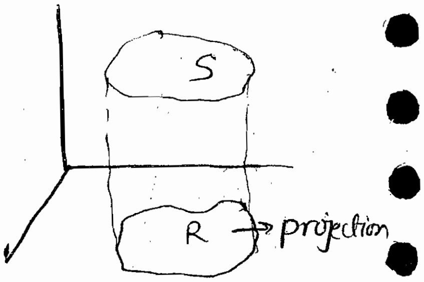
Limits : $\rho = 0$ to $\rho = 2$

$\phi = 0$ to π

$\theta = 0$ to 2π

Surface Integrals

$$\iint_S \bar F\cdot\hat N \, ds$$



Formulas

1) Let R_1 be the projection of S on xy plane.

$$\iint_S \bar F\cdot\hat N \, ds = \iint_{R_1} \bar F\cdot\hat N \frac{dxdy}{|\hat N\cdot\hat k|}, \quad \hat N\cdot\hat k \neq 0$$

2) Let R_2 be projection S on yz plane.

$$\iint_S \bar F\cdot\hat N \, ds = \iint_{R_2} \bar F\cdot N \frac{dydz}{|N\cdot\hat i|}, \quad \hat N\cdot\hat i \neq 0$$

3) Let R_3 be projection of S on xz plane.

$$\iint_S \bar F\cdot\hat N \, ds = \iint_{R_3} \bar F\cdot\hat N \frac{dxdz}{|\hat N\cdot\hat j|}, \quad \hat N\cdot\hat j \neq 0$$

Q:- Find the surface integral of $\iint_S \bar F\cdot\hat N \, ds$. $\bar F = z\bar i + x\bar j - 3y^2 z\bar k$

S: Surface of cylinder, $x^2+y^2=16$ included in 1st ~~order~~ octant b/w $z=0$ & $z=5$.

$f : x^2+y^2 = 16$

$\hat{N} = \frac{\nabla f}{|\nabla f|}$,

$\nabla f = 2x\hat{i} + 2y\hat{j}$

$\hat{N} = \frac{\cancel{2}[x\hat{i}+y\hat{j}]}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x\hat{i}+y\hat{j}}{4}$

on surface $x^2+y^2=16$

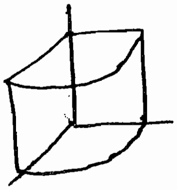
Let R be the projection of S on yz plane.

$${}_S\iint \bar{F}\cdot\hat{N}\,dS = {}_R\iint \bar{F}\cdot\hat{N}\,\frac{dydz}{|N\cdot\hat{i}|}$$

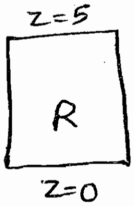
$$\bar{F}\cdot\hat{N} = (z\hat{i}+x\hat{j}-3yz^2\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}}{4}\right)$$

$$= \frac{xz+xy}{4} = \frac{x}{4}(y+z)$$

$$|N\cdot\hat{i}| = \frac{x}{4}$$



Limits of R :—



$x^2+y^2=16$

on yz plane

$x=0,\ y^2=16\ ,\ y=\pm 4$

$y=0$ to 4 & $z=0$ to 5.

$${}_S\iint = \int_0^5\int_0^4 \cancel{\frac{x}{4}}(y+z)\cdot\frac{dydz}{\cancel{x/4}}$$

$$= \int_{z=0}^{5}\int_{y=0}^{4}(y+z)\,dydz$$

$$= 90$$

1) $\nabla r = \vec{r}/r$

2) $\nabla\cdot r = 3$

3) $\nabla\times\bar{r} = \vec{0}$

4) $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

5) $\nabla\cdot(\nabla\phi) = \nabla^2\phi$

Div.(Grad ϕ)

6) $\nabla\times(\nabla\phi) = \vec{0}$

Curl ~~x~~(Grad ϕ)

7) $\nabla\cdot(\nabla\times\bar{F}) = 0$

Div.(Curl F)

8) $\nabla\times(\nabla\times\bar{F}) = \nabla(\nabla\cdot\bar{F}) - \nabla^2(\bar{F})$

(2) $\nabla\cdot r = \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 1+1+1 = 3$

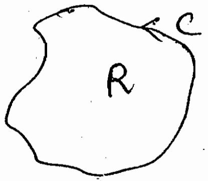
(3) $\nabla\times\vec{r} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \vec{0}$

Integral Theorems :—

- Closed Curve
  - 2D → Green's Theorem
  - 3D ⇒ Stokes Theorem
- Surfaces
  - open
  - closed → Divergence Theorem

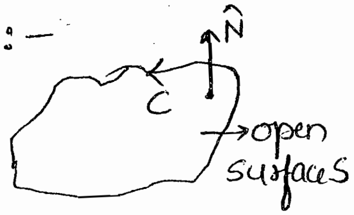
Green's Theorem in a plane —

Let P, Q be differentiable f^ns defined over a region R bounded closed curve C



$$\boxed{_{C} \quad dy = \iint_R \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)dxdy}$$

Stoke's Theorem :-

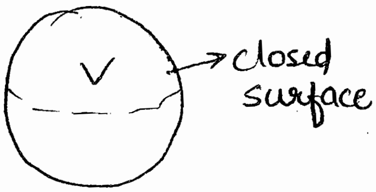


Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be differentiable vector f^n defined over the open surface S bounded by closed ~~surface~~ curve C

$$\boxed{_{C}\oint \bar{F}\cdot\bar{dr} = {}_{S}\iint(\nabla\times F)\cdot\hat{N}\,ds}$$

Gauss Divergence Theorem-

Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be differentiable vector f^n defined over the closed surface S enclosing volume V.



$$\boxed{_{S}\oiint \bar{F}\cdot\hat{N}ds = {}_{V}\iiint(\nabla\cdot\bar{F})dxdydz}$$

Q:- ① $\oiint(\nabla\times\bar{F})\cdot\hat{N}ds =$ ______

Sol^n:- Divergence :

$$_{S}\oiint(\nabla\times\bar{F})\cdot\hat{N}ds = {}_{V}\iiint\nabla\cdot(\nabla\times\bar{F})dV$$
$$=0.$$

② Let F be conservative field.

$$_{C}\oint \bar{F}\cdot\bar{dr} = {}_{S}\iint(\nabla\times\bar{F})\cdot\hat{N}ds$$
$$\bar{F} = \nabla\phi = {}_{S}\iint(\nabla\times\nabla\phi)\cdot\hat{N}ds = 0$$

boundary of open surface S.

$$_{C}\oint \bar{r}\cdot\bar{dr} = {}_{S}\iint(\nabla\times\bar{r})\cdot\bar{N}ds = 0$$

④ $_{S}\oiint \vec{r}\cdot\hat{N}ds =$ ______ where S is Closed surface enclosing volume V.

a) 2V ~~b)~~ 3V c) V d) 0

$$_{S}\oiint \bar{r}\cdot\hat{N}ds = {}_{V}\iiint(\nabla\cdot\bar{r})dxdydz$$
$$= 3\iiint dxdydz$$
$$= 3V.$$

Q:- Find $_{S}\iint \bar{F}\cdot\hat{N}ds$ where S is surface of the sphere $x^2+y^2+z^2=4$

$\bar{F} = (x+z)\bar{i} + (y+z)\bar{j} + (x+y)\hat{k}$

Sol^n:- By Divergence Theorem :-

$$_{S}\oint \bar{F}\cdot\hat{N}ds = {}_{V}\iiint(\nabla\cdot\bar{F})dxdydz$$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(x+y)$$
$$= 1+1+0 = 2$$
$$= 2\iiint dxdydz = 2V$$

Changing into spherical polar co-ordinates :-

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$dxdydz = \rho^2\sin\phi$

Limits :- $\rho = 0$ to 2
$\phi = 0$ to π
$\theta = 0$ to 2π

$$= -2\left[\frac{e^3}{3}\right]_0^2 \left[-\cos\phi\right]_0^{\pi} \left[\theta\right]_0^{2\pi} = \frac{64}{3}\pi \text{ Ans.}$$

$$\boxed{\cos n\pi = (-1)^n}$$

Method 2 :-

$$2\iiint_V dxdydz \quad -① \Rightarrow 2 \text{ Volume of sphere} = 2\cdot\frac{4}{3}\pi r^3 = \frac{8}{3}\pi(2^3) = \frac{64\pi}{3}$$

Q:- $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$. Find $\oiint_S \bar{F}\cdot\hat{N}\,ds$ where S is Surface of cylinder $x^2+y^2=4$, $z=0$ & $z=3$

Soln:- By Gauss Divergence Theorem -

$$\oiint_S \bar{F}\cdot\hat{N}\,ds = \iiint_V (\nabla\cdot\bar{F})\,dxdydz, \quad \nabla\cdot\bar{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 4 - 4y + 2z.$$

$$\Rightarrow \iiint_V (4-4y+2z)\,dxdydz \quad -①$$

Changing cylindrical polar, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $dxdydz = r\,dr\,d\theta\,dz$

Limits :- $r = 0$ to 2, $\theta = 0$ to 2π, $z = 0$ to 3

$$① \equiv \int_0^{2\pi}\int_0^2\int_0^3 \left[4 - 4r\sin\theta + 2z\right] r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[(4-4r\sin\theta)\left[z\right]_0^3 + \left[z^2\right]_0^3\right] r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left[3(4-4r\sin\theta)+9\right] r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[21r - 12r^2\sin\theta\right] dr\,d\theta$$

$$= \int_0^{2\pi} \left[21\left[\frac{r^2}{2}\right]_0^2 - 12\left[\frac{r^3}{3}\right]_0^2 \sin\theta\right] d\theta = \int_0^{2\pi} \left[42 - 32\sin\theta\right] d\theta$$

$$= \int_0^{2\pi} 42\,d\theta - 32\underbrace{\int_0^{2\pi} \sin\theta\,d\theta}_{\substack{\sin(2\pi-\theta) = -\sin\theta \\ \text{Rule 6}}} = 42\left[\theta\right]_0^{2\pi} = 84\pi \text{ Ans.}$$

$$\downarrow$$

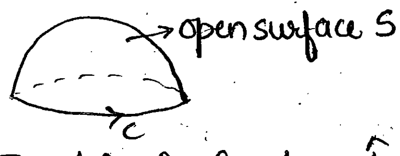
$$0$$

Q:- $\oint_C \left[(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}\right]\cdot\overline{dr}$ where C is the upper half of sphere $x^2+y^2+z^2=1$

$\nearrow$ ∵ it is open surface

Stoke's theorem :- $\oint_C \bar{F}\cdot\overline{dr} = \iint_S (\nabla\times\bar{F})\cdot\hat{N}\,ds$

Soln:-



$$\bar{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$$

$$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \hat{k} \; ; \; \iint_S \hat{k}\cdot\hat{N}\,ds \; ; \; \hat{N} = \frac{\nabla f}{|\nabla f|} \quad \text{Surface } f: x^2+y^2+z^2=1$$

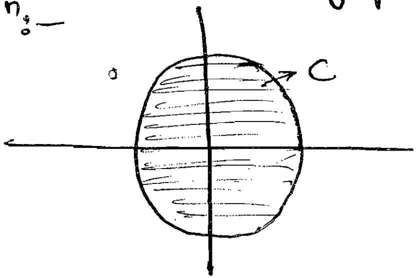
$$\hat{N} = x\hat{i} + y\hat{j} + z\hat{k}.$$

$$_S\iint \bar{K}\cdot\hat{N}\,ds = \iint \bar{K}\cdot\hat{N}\,\frac{dx\,dy}{|\hat{N}\cdot\bar{K}|} = \iint_R dx\,dy$$

$=$ Area of Circle $= \pi r^2 = \pi 1^2 = \pi$ Ans.

Q:- Find $\int_C (x^2+y^2)\,dx + 3xy^2\,dy$ where $C: x^2+y^2=4$ on xy-plane.

Sol^n:-



By Green's theorem :

$$_C\oint P\,dx + Q\,dy = _R\iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

HINT Directly : $x = 2\cos\theta,\ y = 2\sin\theta$
$\theta = 0$ to 2π
$dx = -2\sin\theta\,d\theta,\ dy = 2\cos\theta\,d\theta$

$$_C\int \underbrace{x^2+y^2}_{P}\,dx + \underbrace{3xy^2}_{Q}\,dy$$

$$= _R\iint\left[\frac{\partial}{\partial x}(3xy^2) - \frac{\partial}{\partial y}(x^2+y^2)\right]dx\,dy$$

$$= _R\iint [3y^2 - 2y]\,dx\,dy$$

Using polar coordinates;

$x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$

limit: $r = 0$ to 2 ; $\theta = 0$ to 2π

$$= \int_0^{2\pi}\int_0^2 [3r^2\sin^2\theta - 2r\sin\theta]\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 [3r^3\sin^2\theta - 2r^2\sin\theta]\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[3\left[\frac{r^4}{4}\right]_0^2\sin^2\theta - 2\left[\frac{r^3}{3}\right]_0^2\sin\theta\right]d\theta$$

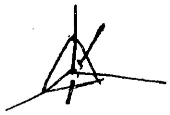
$$= \int_0^{2\pi} 12\sin^2\theta\,d\theta - \frac{16}{3}\underbrace{\int_0^{2\pi}\sin\theta\,d\theta}_{0}$$

$\sin(2\pi-\theta) = -\sin\theta$ Rule6

$= 12\pi$ Ans.

the planes $x+y+z=1,\ x=0,\ y=0$ & $z=0$

$$V = \iiint dx\,dy\,dz$$



$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} dz\,dy\,dx$$

$z=0$, $z = 1-x-y$

$y=0$ to $y=1-x$

$x=0$ to $x=1$

$$= \int_0^1\int_0^{1-x} [z]_0^{1-x-y}\,dy\,dx$$

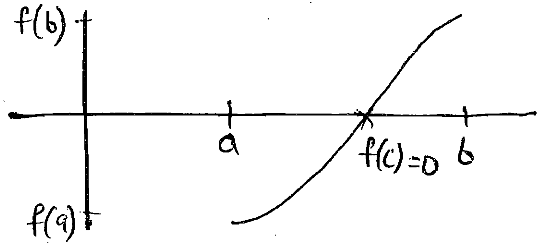
$$= \int_0^1 y - xy - \frac{y^2}{2}\Big|_0^{1-x} dx = \int_0^1 \frac{(1-x)^2}{2}dx$$

$= \frac{1}{6}$ Ans.

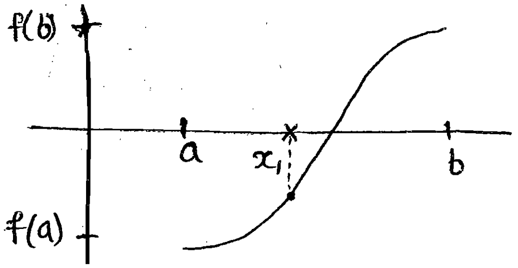
NUMERICAL METHODS

Numerical sol^n to find roots of $f(x)=0$

Result – If $f(a)$ & $f(b)$ are of opposite sign then there exist a root of the eq^n $f(x)=0$ in $[a,b]$



Bisection Method –



Procedure :— To find $f(x)=0$ in $[a,b]$

Let $f(a)<0$ & $f(b)>0$

First approx : $x_1 = \frac{a+b}{2}$

if $f(x_1)=0 \Rightarrow x_1$ is root

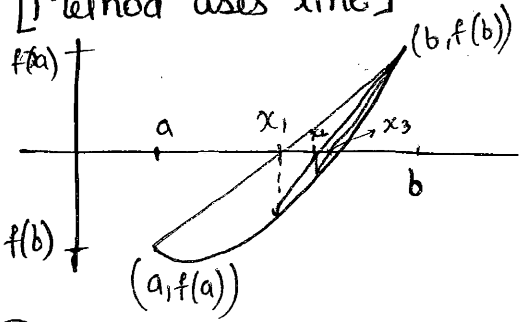
If $f(x_1)<0 \Rightarrow$ Put $a = x_1$

If $f(x_1)>0 \Rightarrow$ Put $b = x_1$

Second approx : $x_2 = \frac{a+b}{2}$

(False Position Method)

[Method uses line]



Procedure :— To find $f(x)=0$ in $[a,b]$

Let $f(a)<0$ & $f(b)>0$

First approx : $x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$

if $f(x_1)=0 \Rightarrow x_1$ is root

If $f(x_1)<0 \Rightarrow$ Put $a=x_1$

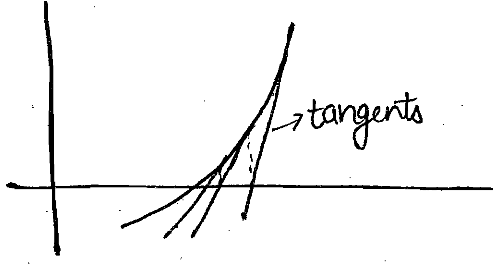
If $f(x_2)>0 \Rightarrow$ Put $b=x_1$

Second approx : $x_2 = \frac{af(b)-bf(a)}{f(b)-f(a)}$

$\vdots$

Newton-Raphson Method :

[Method uses tangents]



Procedure : To find $f(x)=0$.

Let x_0 be initial guess.

First approx : $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Second approx : $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$\vdots$

General Formula

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Bisection	Slow	always conv.	(1)
Regular Falsi	Slow	-u-	(1)
N-R Method	Fast	May diverge	(2)

↑ Order of convergence

Result :-

① → N-R Method does not work : $\boxed{f'(x)=0}$

2) The min^m no. 'n' of iterations reqd. to find root of $f(x)=0$ in $[a,b]$ using bisection method so that error is ϵ is given by

$$\boxed{\frac{|b-a|}{2^n} < \epsilon}$$

Q:— Find root of the eq^n $f(x)=x^3-x-1$ in $[1,2]$ using Bisection method in four stages.

Sol^n:— $f(1)=-1$, $f(2)=5$

	a	b	f(a)	f(b)	$x_i = \frac{a+b}{2}$	$f(x_i)$
①	1	2	-1	5	$\frac{1+2}{2}=1.5$	0.875
②	1	1.5	-1	0.875	1.25	-0.296
③	1.25	1.5	-0.296	0.875	1.375	0.2246
④	1.25	1.375	-0.296	0.2246	$\boxed{1.3125}$ Ans.	

Q:— Min^m no. of iterations reqd. for bisection method in solving the eq^n $x^3+3x-7=0$ in $[1,2]$ so that $|\text{error}| \le 10^{-3}$

Sol^n:— $\epsilon = 10^{-3}$, $a=1$, $b=2$

$$\frac{|b-a|}{2^n} < \epsilon \; ; \; \frac{2-1}{2^n} < 10^{-3}$$

$2^n > 1000 \Rightarrow n > \log_2 1000$

$n > 9.965 \Rightarrow n = 10$. Ans.

Q:— Find roots of the eq^n $f(x) = x^3-4x-9$ in $[2,3]$ using Regular Falsi in 2 stages.

$f(2)=-9$ $f(3)=6$

$f(b) - f(a)$

1. 2   3   −9   6   2.6   = −1.824

2. 2.6   3   −1.824   6 $\boxed{2.69}$ Ans.

Q:- Find $\sqrt{28}$ using N-R method with initial guess $x_0 = 5$

Solⁿ:- $x = \sqrt{28} \Rightarrow x^2 - 28 = 0 = f(x) \Rightarrow f'(x) = 2x$

General Formula : $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 28}{2x_n} = \frac{x_n^2+28}{2x_n}$

$x_{n+1} = \frac{1}{2}\left[x_n + \frac{28}{x_n}\right]$; $x_0 = 5$; $x_1 = \frac{1}{2}\left[5 + \frac{28}{5}\right]$; $x_1 = 5.3$

$x_2 = 5.29$, $x_3 = 5.29$ Ans.

Q:- $f(x) = e^x - 1 = 0$ with initial guess $x_0 = -1$. Solⁿ after one step using N-R Method.

Solⁿ:- $f'(x) = e^x \Rightarrow x_{n+1} = x_n - \frac{f(x)}{f'(x)} = x_0 - \frac{f(x_0)}{f'(x_0)} = -1 - \frac{e^{-1}-1}{e^{-1}} = \overset{2.71828}{e} - 2 = 0.78$

Q:- $f(x) = x^3 - 2x - 5 = 0$ find general formula for N-R Method.

Solⁿ:- $f'(x) = 3x^2 - 2 \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{2x_n^3 + 5}{3x_n^2 - 2}$ Ans.

Numerical Solutions to ODE :-

$$\left.\begin{aligned} \frac{dy}{dx} &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}\right\} \text{Step Size} = \underset{\text{Given.}}{\overset{\downarrow}{h}} \quad \rightarrow \text{Given.}$$

x	y
x_0	y_0 → given.
$x_1 = x_0+h$	$y_1 = y(x_1)$
$x_2 = x_0+2h$	$y_2 = y(x_2)$
⋮	
$x_n = x_0+nh$	$y_n = y(x_n)$

} Find

Euler's Method :-

$$\boxed{y_1 = y_0 + hf(x_0, y_0)}$$

$$y_2 = y_1 + hf(x_1, y_1)$$

Gen. formula:

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

Modified Euler's Method (Runge Kutta Method)
RK method of 2nd order -

$$\boxed{y_1 = y_0 + K}, \quad K = \frac{K_1 + K_2}{2}$$

$K_1 = hf(x_0, y_0)$, $K_2 = hf(x_0+h, y_0+k_1)$

General formula :-

$$\boxed{y_{n+1} = y_n + K} \quad K = \frac{K_1+K_2}{2}$$

$K_1 = hf(x_n, y_n)$

$K_2 = hf(x_n+h, y_n+K_1)$

Runge Kutta Method of order 4

$$\boxed{y_1 = y_0 + K}$$

$$K = \frac{K_1 + 2K_2 + 2K_3 + K_4}{6}$$

$K_1 = hf(x_0, y_0)$; $K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$; $K_4 = hf(x_0+h, y_0+k$

$y_{n+1} = y_n + K$

$K = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

$k_1 = hf(x_n, y_n)$, $k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$

$k_4 = hf(x_n + h, y_n + k_3)$.

Q:- $\frac{dy}{dx} - y = 0$ $y(0) = 0$ using Euler's Method with step size $0.1 = h$ Find $y(0.3)$

Sol^n:- $\frac{dy}{dx} = x + y \Rightarrow f(x, y) = x + y$

x	y
$x_0 = 0$	$y_0 = 0$
$x_1 = 0.1$	$y_1 = y(0.1) = 0$
$x_2 = 0.2$	$y_2 = y(0.2) = 0.01$
$x_3 = 0.3$	$y_3 = y(0.3) = 0.031$

$y_1 = y_0 + hf(x_0, y_0) = 0 + 0.1 \times 0 = 0$

$y_2 = y_1 + hf(x_1, y_1) = 0 + 0.1 f(0.1, 0) = 0.01$

$y_3 = y_2 + hf(x_2, y_2) = 0.031$

Q:- $\frac{dy}{dx} = 1 + y^2$; $y(0) = 0$ using R-K method of 4th order with step size 0.2, find $y(0.2)$

Sol^n:- $f(x, y) = 1 + y^2$, $h = 0.2$

x	y
$x_0 = 0$	$y_0 = y(0) = 0$
$x_1 = 0.2$	$y_1 = y(0.2) =$

$y_1 = y_0 + K$

$K = \frac{K_1 + 2K_2 + 2K_3 + K_4}{6}$

$K_1 = hf(x_0, y_0) = 0.2 f(0, 0) = 0.2 \times 1 = 0.2$

$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) = 0.2 f(0.1, 0.1)$
$= 0.2[1 + 0.1^2] = 0.202$

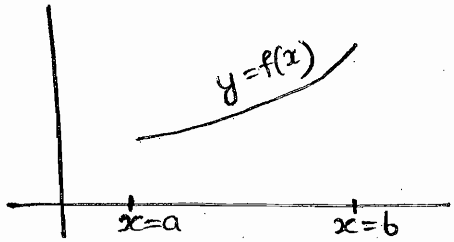
$= 0.20204$

$K_4 = hf(x_0 + h, y_0 + K_3) = 0.20816$

$K =$ ~~0.135~~ 0.2027

$y_1 = y_0 + K = 0.2027$

Length of arc :-



Length of arc : $y = f(x)$ b/w $x = a$ to $x=$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

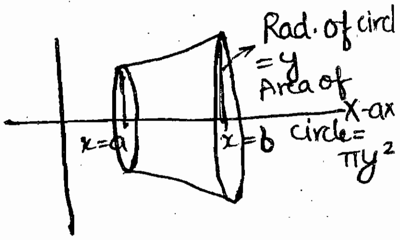
Length of arc : $x = g(y)$ b/w $y = c$ to $y = d$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Volume of solid of revolution -

1) The volume of solid of revolution by revolving $y = f(x)$ b/w $x = a$ to $x = b$ around x-axis

$$V = \int_a^b \pi y^2 dx$$



2) The volume of solid of revolution by revolving $x = g(y)$ b/w $y = c$ to $y = d$ around y-axis

$$V = \int_c^d \pi x^2 dx$$

Find length of arc ;

Sol^n:- $\dfrac{dy}{dx} = \dfrac{2}{3} \times \dfrac{3}{2} x^{1/2} = x^{1/2}$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1+x}\, dx$$

$$= \frac{2}{3}\left[2\sqrt{2} - 1\right] \text{ Ans.}$$

Q:- Parabolic arc: $y = \sqrt{x}$ $1 \le x \le 2$ is revolved arnd x-axis. Volume of solid of revolution.

$$V = \int_1^2 \pi y^2 dx = \int_1^2 \pi x\, dx$$

$$= \frac{3\pi}{2} \text{ Ans.}$$

Gamma f^n :-

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma(1) = 1 \;, \; \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(n+1) = \begin{cases} n! & n \text{ is integer} \\ n\Gamma(n) & \text{o/w.} \end{cases}$$

$$\Gamma(3) = 2! = 2$$

$$\Gamma(5) = 4! = 24$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(\tfrac{1}{2}+1\right) = \tfrac{1}{2}\Gamma\tfrac{1}{2} = \tfrac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\sqrt{\pi}$$

$$\Gamma\left(\tfrac{7}{2}\right) = \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\sqrt{\pi}$$

$$\int_{-\infty} \underbrace{\qquad}_{\text{even } f^n} dx$$

$$= 2\int_0^\infty e^{-x^2} dx \quad ; \; x^2 = t \quad 2x\,dx = dt$$

$x = 0 \; ; \; t = 0$

$x \to \infty \; ; \; t \to \infty$

$$= 2\int_0^\infty e^{-t} \frac{dt}{2(t)^{1/2}} = \int_0^\infty e^{-t} t^{1/2-1} dt$$

$$= \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Q:- $\displaystyle\int_0^\infty e^{-y^3} y^{1/2} dy$

$y^3 = t$

$3y^2 dy = dt$

$dy = \dfrac{dt}{3(y)^2} = \dfrac{dt}{3t^{2/3}}$

Limits

$y = 0 \Rightarrow t = 0$

$y \to \infty \Rightarrow t \to \infty$

$$\int_0^\infty e^{-t} t^{1/6} \frac{dt}{3t^{2/3}} = \frac{1}{3}\int_0^\infty e^{-t} t^{\frac{1}{6}-\frac{2}{3}} dt$$

$$= \frac{1}{3}\int_0^\infty e^{-t} t^{1/2-1} dt = \frac{1}{3}\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{3}$$

LAPLACE TRANSFORM

$$\boxed{L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt}$$

1) $L\{1\} = 1/s$

2) $L\{e^{at}\} = \dfrac{1}{s-a}$

3) $L\{\sinh at\} = \dfrac{a}{s^2-a^2}$

4) $L\{\cosh at\} = \dfrac{s}{s^2-a^2}$

$s-ia$

$$L\{\cos at + i\sin at\} = \frac{s+ia}{s^2+a^2}$$

5) $L\{\cos at\} = \frac{s}{s^2+a^2}$

6) $L\{\sin at\} = \frac{a}{s^2+a^2}$

7) $L\{t^n\} = \int_0^\infty e^{-st} t^n dt$

Put $st = x$ **Limit**

$s\,dt = dx$ $t=0 \quad x=0$

$t \to \infty \quad x \to \infty$

$$= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{(n+1)-1} dx$$

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$$

Q:- $L\{t\} = \frac{1}{s^2}$

$$L\{t^2\} = \frac{2}{s^3}$$

$$L\{t^6\} = \frac{\Gamma(7)}{s^7} = \frac{6!}{s^7}$$

$$L\{t^{1/2}\} = \frac{\Gamma(\frac{3}{2})}{s^{3/2}} = \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}}$$

$$L\{t^{3/2}\} = \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}}$$

$$L\{t^{5/2}\} = \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{7/2}}$$

$$L\{t^{9/2}\} = \frac{\frac{9}{2}\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{11/2}}$$

Unit Step f^n :—

$$u(t-a) = \begin{cases} 0 & t<a \\ 1 & t \geq a \end{cases}$$

$a \qquad -s \,\Big|_a$

$$= \frac{e^{-as}}{s}$$

Unit Impulse F^n:-

$$\delta(t-a) = \begin{cases} \frac{1}{\varepsilon} & a<t<a+\varepsilon \\ 0 & o|w \end{cases} \qquad \boxed{\varepsilon \to 0}$$

$\downarrow$

Dirac Delta F^n

$$L\{\delta(t-a)\} = e^{-as}$$

L.T. of periodic f^ns :— of period T

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$$

L.T. of $f(t)$ of period 2π

$$f(t) = \begin{cases} \sin t & 0<t<\pi \\ 0 & \pi \leq t \leq 2\pi \end{cases}$$

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}}\left[\int_0^\pi e^{-st}\sin t\,dt + \int_\pi^{2\pi} e^{-st}\,0\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\int_0^\pi e^{-st}\sin t dt\right]$$

Remember –

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-s\pi}}{s^2+1}\left[-s\cdot 0-(-1)\right] - \frac{e^0}{s^2+1}\left[-s\cdot 0-1\right]\right]$$

$$= \frac{1}{(1-e^{-2\pi s})(s^2+1)}\left[e^{-s\pi}+1\right]$$

$$= \frac{e^{\pi s}}{(e^{\pi s}-1)(s^2+1)}$$

$$L\{f(t)\} = F(s)$$

$$L\{e^{at} f(t)\} = F(s-a)$$

1) $L\{e^{at} t^n\} = \dfrac{\Gamma(n+1)}{(s-a)^{n+1}}$

2) $L\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$

3) $L\{e^{at}\cos bt\} = \dfrac{(s-a)}{(s-a)^2+b^2}$

$$(s-2)^3\}$$

$$= L^{-1}\left\{\frac{((s-2)+2)^2}{(s-2)^3}\right.$$

$$= e^{2t} L^{-1}\left\{\frac{(s+2)^2}{(s)^3}\right\} = e^{2t} L^{-1}\left\{\frac{s^2+4s+4}{s^3}\right\}$$

$$= e^{2t}\left[L^{-1}\frac{1}{s} + 4L^{-1}\frac{1}{s^2} + 4L^{-1}\frac{1}{s^3}\right]$$

$$= e^{2t}\left[1 + 4t + \frac{4t^2}{2}\right.$$ Ans.

$L\{f(t)\} = F(s)$	$L^{-1}\{F(s)\} = f(t)$
(1) $L\{1\} = 1/s$	$L^{-1}\{1/s\} = 1$
(2) $L\{t\} = 1/s^2$	$L^{-1}\{1/s^2\} = t$
(3) $L\{e^{at}\} = 1/s-a$	$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
(4) $L\{\sinh at\} = \frac{a}{s^2-a^2}$	$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$
(5) $L\{\cosh at\} = \frac{s}{s^2-a^2}$	$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$
(6) $L\{\sin at\} = \frac{a}{s^2+a^2}$	$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$
(7) $L\{\cos at\} = \frac{s}{s^2+a^2}$	$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$
(8) $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$	$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{\Gamma(n)}$

First Translation of ILT :- $\boxed{L^{-1}\{F(s-a)\} = e^{at} L^{-1}\{F(s)\}}$

Q :- $L^{-1}\left\{\dfrac{s+2}{s^2+4s+9}\right\} = L^{-1}\left\{\dfrac{s+2}{s^2+4s+4+5}\right\} = L^{-1}\left\{\dfrac{s+2}{(s+2)^2+5}\right\}$

$$= e^{-2t} L^{-1}\left\{\frac{s}{s^2+(\sqrt{5})^2}\right\} = e^{-2t}\cos\sqrt{5}t$$

(Derivative of transform) –

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Ex :– $L\{t\cos t\}$

$$L\{\cos t\} = \frac{s}{s^2+1}$$

$$L\{t\cos t\} = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$$

$$= \left[\frac{s^2-1}{(s^2+1)^2}\right]$$

Ex :- $L\{t^2 \sin at\}$

$$L\{e^{iat} t^2\}$$

$$L\{t^2\} = \frac{2}{s^3}$$

$$L\{e^{iat} \cdot t^2\} = \frac{2}{(s-ia)^3}$$

$$L\{t^2 \sin at\} = \text{Im}\ \frac{2(s+ia)^3}{(s-ia)^3(s+ia)^3}$$

$$L\{t^2 \sin at\} = \frac{6s^2a - 2a^3}{(s^2+a^2)^3}$$ Ans.

Division by t [Integral L.T.]

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

(1) $L\left\{\frac{\sin t}{t}\right\}$; $L\{\sin t\} = \frac{1}{s^2+1}$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \tan^{-1}s \Big|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

Q :- $\int_0^\infty \frac{\sin t}{t}\,dt$

$$L\left\{\frac{\sin t}{t}\right\} = \cot^{-1}s$$

$$\int_0^\infty e^{-st}\frac{\sin t}{t}\,dt = \cot^{-1}s$$

$$\int_0^\infty \frac{\sin t}{t}\,dt = \cot^{-1}0 = \pi/2$$

Ex :– $L\left\{\frac{e^{at}}{t}\right\}$

$$L\{e^{at}\} = \frac{1}{s-a}\ ;\ L\left\{\frac{e^{at}}{t}\right\} = \int_s^\infty \frac{1}{s-a}\,ds$$

$$= \log(s-a)\Big|_s^\infty$$ It does not exist.

Ex :– $L\left\{\frac{e^{at}-e^{bt}}{t}\right\}$

$$L\{e^{at}-e^{bt}\} = \frac{1}{s-a} - \frac{1}{s-b}$$

$$L\left\{\frac{e^{at}-e^{bt}}{t}\right\} = \int_s^\infty \left[\frac{1}{s-a} - \frac{1}{s-b}\right] ds$$

$$= \Big[\log(s-a) - \log(s+b)\Big]_s^\infty = \log\frac{s-a}{s-b}\Big]_s^\infty$$

$$\lim_{s\to\infty} \log\left(\frac{s-a}{s-b}\right) - \log\left(\frac{s-a}{s-b}\right)$$

$$\lim_{s\to\infty} \log\left(\frac{1-a/s}{1-b/s}\right) - \log\left(\frac{s-a}{s-b}\right) = 0 - \log\left(\frac{s-a}{s-b}\right)$$

$$= \log\left(\frac{s-b}{s-a}\right)$$

L.T. of derivative (Multiplication by s):-

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$L\{f'''(t)\} = s^3F(s) - s^2F(0) - sf'(0) - f''(0)$$

Ex :- $y'' + 4y = 12t$; $y(0) = 0$; $y'(0) = 9$

Sol^n :– $L\{y\} = Y(s)$

$$L\{y'' + 4y\} = 12\,L\{t\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{12}{s^2}$$

$$[s^2+4]Y(s) - 9 = \frac{12}{s^2}$$

$$Y(s) = \frac{12 + 9s^2}{s^2(s^2+4)} = \frac{9s^2 + 36 - 36 + 12}{s^2(s^2+4)}$$

$$= \frac{9(s^2+4)}{s^2(s^2+4)} - \boxed{\frac{24}{s^2(s^2+4)}} \rightarrow 24\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right]$$

$$= \frac{9}{s^2} - \frac{6}{s^2} + \frac{6}{s^2+4} = \frac{3}{s^2} + \frac{6}{s^2+4}$$

$y(t) = 3t + 3\sin 2t$.

$F(s) = \log(s-3) - \log(s-2)$

$F'(s) = \frac{1}{s-3} - \frac{1}{s-2}$

$L^{-1}\{F'(s)\} = e^{3t} - e^{2t}$

$-tf(t) = e^{3t} - e^{2t}$

$f(t) = \frac{e^{2t} - e^{3t}}{t}$ Ans.

The end of math.